Robust Solution Method for the Customer Choice Cancellation Model[I]

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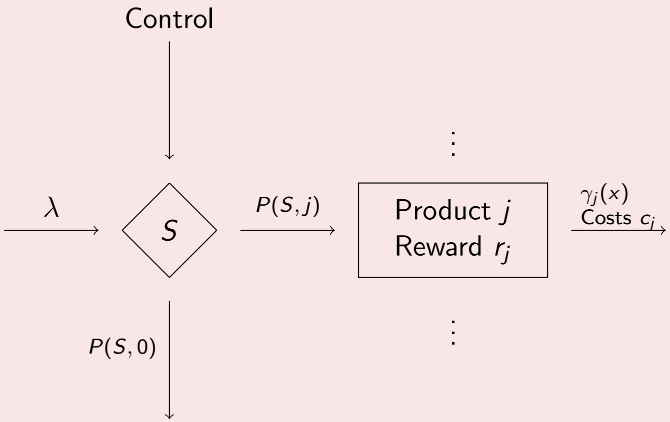
Introduction

- Problem description
- Brief introduction to robust optimisation
- Revenue management & robust optimisation
- Numerical results

Problem

- Most solution methods optimise *nominal* solution
- Solution methods don't take into account *uncertainty* in parameters.



Solution: Robust Optimisation

Optimise **worst case scenario**. E.g., consider a linear program:

$$\min\left\{c^\top x\,|Ax \leq b\right\},$$

with $c \in \mathbb{R}^n$ the cost vector, $x \in \mathbb{R}^n$ the vector of decision variables, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. The robust counterpart is given by

$$\min\left\{c^\top x\,|Ax \leq b, A \in Z\right\},$$

with $A \in \mathbb{R}^{m\times n}$ the uncertain parameters in *uncertainty set* $Z \subset \mathbb{R}^{m\times n}$.
The uncertainty can be approached constraint-wise:

$$(a + B\zeta)^\top x \leq b, \quad \forall \zeta \in Z,$$

with the nominal value $a \in \mathbb{R}^n$ constant.

Example: ellipsoidal/ball uncertainty

Consider the linear constraint:

$$(a + B\zeta)^\top x \leq b, \quad \forall \zeta \in Z,$$

with the uncertainty set Z given by

$$Z = \{\zeta \in \mathbb{R}^m \,|\, \|\zeta\|_2 \leq \rho\}.$$

Then the robust counterpart is given by

$$a^\top x + \rho \|B^\top x\|_2 \leq b.$$

ϕ-divergence uncertainty sets (Ben-tal et alii, 2013)

Consider a function $\phi \colon \mathbb{R} \to \mathbb{R}$ that is convex for $t \geq 0$, $\phi(1) = 0$, $0\phi(1/0) := a \lim_{t\to\infty} \phi(t)/t$ for $a > 0$, and $0\phi(0/0) := 0$. The **ϕ-divergence measure** $I_\phi(p, q)$ between two vectors p and q, $p, q \in \mathbb{R}^n$, is defined as

$$I_\phi(p, q) := \sum_{i=1}^{n} q_i \phi(p_i/q_i).$$

Theorem 4.1 in Ben-Tal et alii (2013)

Consider the linear constraint

$$(a + Bp)^\top x \leq b, \qquad\qquad p \in Z, \tag{1}$$

where $p \in \mathbb{R}^m$ is the uncertain parameter, and

$$Z = \left\{ p \in \mathbb{R}^m \,|\, p \geq 0, Cp \leq d, I_\phi(p,q) \leq \rho \right\},$$

is the uncertainty region of p with $q \in \mathbb{R}^m_+$, $\rho > 0$, $d \in \mathbb{R}^k$, and $C \in \mathbb{R}^{k \times m}$. Then a vector $x \in \mathbb{R}^n$ satisfies (1) if and only if there exist $\eta \in \mathbb{R}^k$ and $\xi \in \mathbb{R}$ such that (x, η, ξ) satisfies

$$\begin{cases} a^\top x + d^\top \eta + \rho\xi + \xi \sum_i q_i \phi^* \left(\dfrac{B_i^\top x - C_i^\top \eta}{\xi} \right) \leq b, \\ \eta \geq 0, \xi \geq 0, \end{cases}$$

where B_i and C_i are the i-th columns of B and C, respectively, and ϕ^* is the conjugate function of ϕ.

Remark

ϕ-divergence uncertainty sets can be applied for any probabilities. In particular, they can be applied to any **choice model**. For example, consider the assortment problem:

- n types of products $N = \{1, \ldots, n\}$ available
- $S \subset N$ needs to be displayed s.t. revenue is maximised
- When $S \subset N$ is displayed the probability that product $j \in S$ is purchased is $P_j(S)$.

Example

The *variation distance* is given by

$$\phi(t) = |t - 1|$$

The ϕ-divergence of the variation distance is given by

$$I_\phi(p, \bar{p}) = \sum |p_i - \bar{p}_i|.$$

This is the ℓ_1 norm. Larger deviation is punished linearly more than smaller deviations.

Uncertain parameters in Bellman equation of dynamic program

$$V_t(y) = \max_{S \subset N} \left\{ \sum_{j \in S} \tilde{P}_j(S) \left[\tilde{r}_j + V_{t-1}(y+1) \right] + \tilde{P}_{n+1}(S) V_{t-1}(y-1) + \tilde{P}_0(S) V_{t-1}(y) \right\}$$

Robust reformulation

The uncertainty set Z_S is then given by

$$Z_S = \left\{ p \in \mathbb{R}^{n+2} \,|\, p \geq 0, Cp \leq d, I_\phi(p, \bar{p}) \leq \rho \right\},$$

with

$$C_{\bullet j} = \begin{cases} (1, -1) & \text{if } j \in S \cup \{0, n+1\}, \\ (0, 0) & \text{otherwise.} \end{cases}$$

$$d = (1, -1).$$

Robust reformulation

The nominal dynamic program under uncertainty sets Z_S can be solved using the following recursive formula (see Nilim and Ghaoui, 2005):

$$V_t(y) = \max_{S \subset N} \left\{ \min_{p \in Z_S} \left\{ \sum_{j \in S} p_j [\tilde{r}_j + V_{t-1}(y+1)] + p_{n+1} V_{t-1}(y-1) + p_0 V_{t-1}(y) \right\} \right\}.$$

The uncertainty problem that needs to be solved is

$$\min \left\{ t \in \mathbb{R} \,\middle|\, \sum_{j \in S} p_j [\tilde{r}_j + V_{t-1}(y+1)] + p_{n+1} V_{t-1}(y-1) + p_0 V_{t-1}(y) - t \leq 0, \forall p \in Z_S \right\}.$$

Robust reformulation

Define $x := (t, x_0)$, $a := (-1, 0)$, and

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ V_{t-1}(y) & \tilde{r}_1 + V_{t-1}(y+1) & \cdots & \tilde{r}_n + V_{t-1}(y+1) & V_{t-1}(y-1) \end{pmatrix}.$$

Then the constraint can be rewritten to

$$\begin{cases} 0 & \geq (a + Bp)^\top x, \ \forall p \in Z_S, \\ x_0 & = 1. \end{cases}$$

The robust counterpart is given by

$$\begin{aligned} a^\top x + d^\top \eta + \rho_P \xi + \xi \sum_{i=0}^{n} \bar{p}_i \phi^* \left(\frac{B_i^\top x - C_i^\top \eta}{\xi} \right) &\leq 0, \\ \eta &\geq 0, \\ \xi &\geq 0. \end{aligned}$$

Example

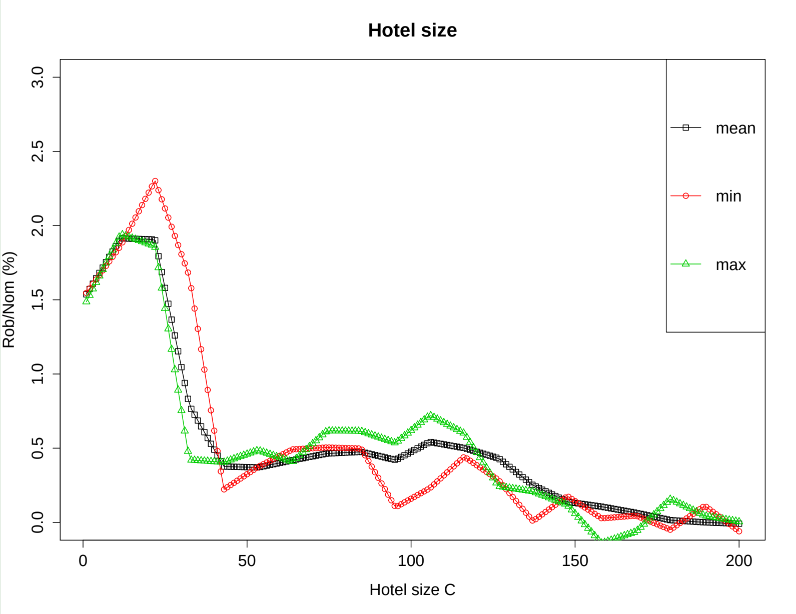
A tractable reformulation for the variation distance is given by

$$\begin{cases} a^\top x + \eta_1 - \eta_2 + \xi \rho_P + \bar{p}^\top y \leq 0, & \\ y_i \geq -\xi, & \forall i, \\ y_i \geq B_i^\top x - \eta_1 + \eta_2, & \forall i, \\ B_i^\top x - C_i^\top \eta \geq \xi, & \forall i, \\ \eta \geq 0, \xi \geq 0. & \end{cases}$$

Numerical results

- Hotel with C rooms
- Booking horizon is T time units
- $n = 10$ different prices + conditions available
- Two segments: Segment 1 sensitive to price; Segment 2 less sensitive to price
- Compare nominal solution to robust solution, using *estimated* parameters

Numerical results



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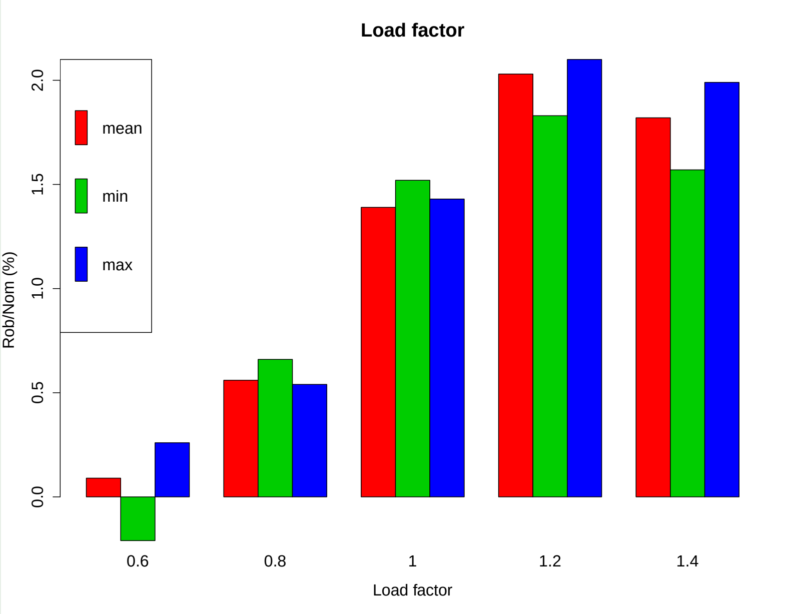
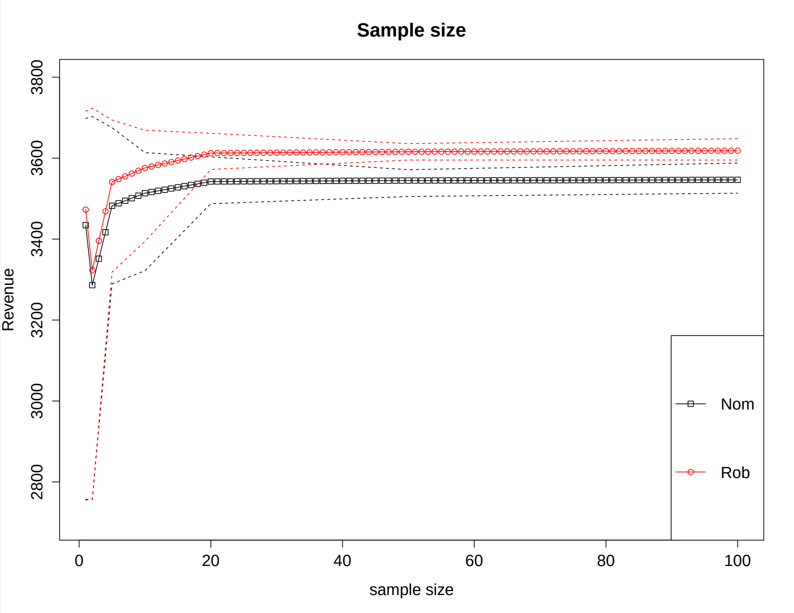
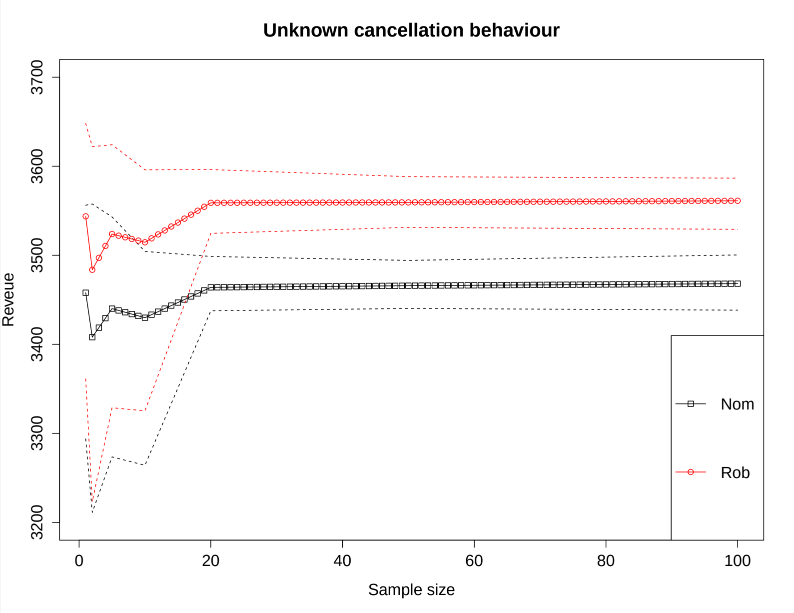


Numerical results



Numerical results



Conluding remarks

- Tractable robust solution methods exist for any choice-model
- Using robust solution method may increase revenue
- Robust solution method performs better for smaller hotels than for larger hotels
- Future direction: which ϕ-divergence measure performs best?
- Future direction: extend model to network problem (multiple nights/flight legs)