Pricing-based revenue management for flexible products on a network

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Author biography

Dirk Sierag is a mathematician with special interest in operations research and financial mathematics. He holds a PhD position in dynamic pricing and revenue management at the Center for Mathematics and Computer Science (CWI), Amsterdam. His research emphasizes on customer choice behavior, (online) ratings and reviews, overbooking and cancellations, and derivative pricing.

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Abstract

This paper proposes and analyses a pricing-based revenue management model that allows flexible products on a network, with a non-trivial extension to group reservations. Under stochastic demand the problem can be solved using dynamic programming, though it suffers from the curse of dimensionality. The solution under deterministic demand gives an upper bound on the stochastic problem, and serves as a basis for two heuristics, which are asymptotically optimal. Numerical studies, based on a problem instance from practice, show that the heuristics perform well, even under uncertainty in demand. Moreover, neglecting flexible products can lead to substantial revenue loss.

Keywords: Revenue Management, dynamic pricing, flexible products, group bookings, network revenue managment

1 Introduction

In line with Gallego and Phillips (2004), a *flexible product* is a set of alternatives offered by a firm such that the customer is assigned to one of the alternatives at a later point in time. Offering flexible products is common practice in certain industries, such as digital and TV advertising, with 2014 global market volumes of US \$146.6 and \$189.4 billion, respectively, and growing (McKinsey&Company, 2015). Research strongly suggests that offering flexible products, alongside traditional *specific products*, can achieve better price discrimination a potentially increase revenue in other application areas as well, such the hotel industry and aviation industry (Gallego et al., 2004; Gallego and Phillips, 2004; Mang et al., 2012). Offering the right price at the right time is crucial for the success of flexible products. However, no *pricing-based* revenue management (RM) model with flexible products has yet been studied in literature.

This paper proposes and analyses a *pricing-based* RM model that allows flexible products on a network with group reservations. A stochastic model with multiple time periods is proposed, where demand is Poisson distributed and time-dependent. Due to the fact that flexible products are allocated to specific products at some point in time, the modelling of group reservations of flexible products differs from traditional group reservations for specific products. A dynamic programming formulation is given to solve the problem, though it suffers from the curse of dimensionality. The deterministic variant of the problem is tractable and provides an upper bound on the stochastic problem. Also, the deterministic pricing strategy serves as a basis for two asymptotically optimal heuristics.

An analysis of a pricing-based approach, as opposed to quantity-based, is desired for two reasons. First, in some industries, like retailing, pricing is common practice and makes more sense than capacity allocation. Second, pricing models in essence combine pricing decisions with capacity allocation decisions. If one wants to allocate capacity, then this can also be done by setting the price high enough. Not only will this give the desired capacity allocation, but it will also lead to higher revenues.

This article builds on two main research areas. The first area is pricing-based RM. Of particular interest to this paper is the excellent study on network RM pricing by Gallego and Van Ryzin (1997). The authors propose two heuristics to solve the intractable pricing model, on which the heuristics of this paper are based. Improved solution methods that deal with the intractability of network models can be found in Maglaras and Meissner (2006), who introduced an action-space reduction algorithm, and, more recently, Zhang and Lu (2013), where a resource decomposition approach is considered. See Bitran and Caldentey (2003) and Soon (2011) for an overview of relevant pricing literature.
The second research area studies flexible products. The concept of flexible products was introduced in the seminal paper by Gallego et al. (2004), who argue in favour of its applicability in many industries and analyse a model with one flexible product on two specific products. Subsequently Gallego et al. (2004) proposed and analysed a quantity-based network RM model with flexible products, together with an extension to choice-based demand. More recently, Petrick et al. (2010) provide an overview of different quantity-based RM methods; Petrick et al. (2012) study the effect of flexible products on revenue when demand is uncertain due to forecast errors; and Gönsch et al. (2014) provide an improved DLP approach to the flexible network problem. The empirical study by Mang et al. (2012) shows by means of a large field study of a low-cost airline that flexible products can increase profit by 5%. Related to flexible products is the study of probabilistic goods by Fay and Xie (2008), where capacity is allocated to the product immediately after purchase, rather than at a later moment in time.

This paper makes the following research contributions: 1) a pricing-based network RM model that incorporates flexible products and group reservations; 2) performance bounds and asymptotically optimal solution methods to the intractable stochastic problem; 3) numerical studies, motivated by a problem instance from industry, underline the effectiveness of flexible products and the performance of the heuristics, also under uncertainty in demand.

The remainder of this paper is organised as follows. First, in Section 2, the pricing-based network RM model under stochastic and deterministic demand is introduced. Also, the extension to group bookings is presented, as well as an upper bound on the stochastic problem. Second, Section 3 proposes two asymptotically optimal heuristics based on the deterministic problem to solve the intractable stochastic problem. Third, Section 4 provides numerical results the effects of flexible products, the performance of the heuristics, and the effects of uncertainty in demand on revenue. Finally, Section 5 concludes the paper.

2 Model description

In this section both a stochastic and a deterministic pricing model for flexible products on a network are introduced. The curse of dimensionality prevents the problem to be solved exactly. The solution to the deterministic problem is an upper bound on the stochastic solution, and is used in the heuristics that are presented in Section 3. Finally, an extension to a non-trivial model for group reservations is introduced.

2.1 Stochastic model

Consider a firm that has m types of perishable *resources* available with capacity vector $C \in \mathbb{N}^m$, which can only be used in T time units. The firms sells the resources by offering n *specific products* $N = \{1, \ldots, n\}$ and f *flexible products* $F = \{1, \ldots, f\}$, which consume one or more resources. The resource consumption for the specific products is described by the incidence matrix $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, where $a_{ij} = 1$ if product j consumes resource i. A flexible product $k \in F$ consists of a set of alternatives $F_k \subset N$ of specific products. At time T each customer that purchased a flexible product k is assigned to one of the $f_k = |F_k|$ alternatives $j \in F_k$.

The booking horizon is divided into T time periods in which products are offered. Demand for products in time period t is Poisson distributed with parameter $\lambda_j(p, t)$ for specific products $j \in N$ and parameter $\gamma_k(p,t)$ for flexible products $k \in F$, where $p \in \mathbb{R}_+^{n+f}$ is the price vector of the products. The demand functions $\lambda(p,t)$ and $\gamma(p,t)$ satisfy the following *regularity conditions*:

1. The demand function $(\lambda(p,t), \gamma(p,t))$ has an inverse $p(\lambda, \gamma, t)$. Hence the demand can and will function as the decision variables.

2. The *revenue* or *reward* at time t defined by $r(\lambda, \gamma, t) = (\lambda, \gamma)^\top p(\lambda, \gamma, t)$ is continuous, bounded, and concave.

3. The following asymptotic results hold for all finite (λ, γ) and for all $j \in N$ and $k \in F$:

$$\lim_{(\lambda,\gamma):\, \lambda_j \to 0} \lambda_j p_j(\lambda, \gamma, t) = 0,$$
$$\lim_{(\lambda,\gamma):\, \gamma_k \to 0} \gamma_k p_k(\lambda, \gamma, t) = 0.$$

4. The following ensures the revenue is bounded:

$$\sup_{j,t}[\arg \max_{(\lambda,\gamma):\, \lambda_j(t)} \lambda_j(t) p_j(\lambda, \gamma, t)] < \infty,$$
$$\sup_{k,t}[\arg \max_{(\lambda,\gamma):\, \gamma_k(t)} \gamma_k(t) p_k(\lambda, \gamma, t)] < \infty.$$

5. There exist null prices $p_j^\infty(t)$ and $p_k^\infty(t)$ such that:

   (a) if $\{p^i\}$ is a sequence for which $p_j^i \to p_j^\infty$, then $\lim_{i\to\infty} \lambda_j(p^i, t) = 0$,

   (b) if $\{p^i\}$ is a sequence for which $p_k^i \to p_k^\infty$, then $\lim_{i\to\infty} \gamma_k(p^i, t) = 0$.

A function that satisfies these conditions is called *regular*.

Let $v \in \mathbb{N}$ be the number of bookings for specific products. Let $s = C - Av$ be the capacity not yet committed. Let $y \in \mathbb{R}^f$ be the number of accepted requests for flexibles. Define (s, y) as the state of the network. Then (s, y) is feasible if and only if it satisfies the following system of linear (in)equalities

$$
\begin{aligned}
s &\geq 0, \\
\sum_{k=1}^{f} B_k z_k &\leq s, \\
y_k - \mathbf{1}^\top z_k &= 0, \quad \forall k,
\end{aligned}
\tag{1}
$$

where B_k is the submatrix of A containing columns corresponding to the products in F_k, and $z_k \in \mathbb{Z}^{f_k}$ represents the allocation of flexible products to specific products. Define $B = (B_1 \cdots B_f)$ and $z = (z_1, \ldots, z_f)$. Define $U \in \mathbb{R}^{f \times |z|}$ by $u_{ij} = 1$ if j corresponds to flexible product i. Let $N(t)$ and $M(t)$ be the random Poisson demand under $\lambda(t)$ and $\gamma(t)$. The objective is to maximise expected revenue:

$$
\begin{aligned}
E\left[\sum_{t=1}^{T} r(t, \lambda, \gamma)\right] =& E\left[\sum_{t=1}^{T} \big(N(t), M(t)\big)^\top p(t, \lambda, \gamma)\right] \\
=& E\left[\sum_{t=1}^{T} \left[\sum_{j \in N} N_j(t) p_j(t, \lambda, \gamma) + \sum_{k \in F} M_k(t) p_k(t, \lambda, \gamma)\right]\right].
\end{aligned}
\tag{2}
$$

Demand has to satisfy the feasibility constraints (1). The state (s, y) has to be feasible in each time period, but the flexible products can be distributed over their corresponding specific products at the end of the booking horizon. Therefore it is not necessary to use a dummy variable vector z for each time period, but only one for the whole system. This leads to the following feasibility constraints:

$$
\begin{aligned}
\sum_{t=1}^{T} AN(t, \lambda(t), \gamma(t)) + Bz &\leq C, \\
\sum_{t=1}^{T} M(t, \lambda(t), \gamma(t)) - Uz &= 0.
\end{aligned}
\tag{3}
$$

The stochastic optimisation problem is equal to

$$\max \left\{ E\left[\sum_{t=1}^{T} \big(N(t), M(t)\big)^\top p(t, \lambda, \gamma) \right] \middle| \begin{array}{ll} \sum_{t=1}^{T} AN(t) + Bz & \leq C \\ \sum_{t=1}^{T} M(t) - Uz & = 0 \\ \lambda, \gamma, z & \geq 0 \\ z \text{ integer} & \end{array} \right\}. \tag{4}$$

To deal with the stochasticity of demand, consider the following set-up to solve the problem. Assume that time periods are small enough that the probability that more than one purchase occurs in one time period is very small. The probability that no purchase occurs in time period t is then equal to

$$1 - \sum_{j \in N} \lambda_j(p, t) - \sum_{k \in F} \gamma_k(p, t).$$

The objective is to maximise revenue to go $V_t(s, y)$. The Bellman equation equals

$$\begin{aligned} V_t(s, y) = \max_{p \in \mathbb{R}^{n+f}} \Bigg\{ & \sum_{j \in N} \lambda_j(p, t)[p_j + V_{t+1}(s - A_j, y)] \\ & + \sum_{k \in F} \gamma_k(p, t)[p_k + V_{t+1}(s, y + e_k)] \\ & + \left(1 - \sum_{j \in N} \lambda_j(p, t) - \sum_{k \in F} \gamma_k(p, t)\right) V_{t+1}(s, y) \Bigg\}. \end{aligned} \tag{5}$$

This problem is intractable for all practical purposes due to the curse of dimensionality. Therefore approximations or heuristics need to be considered. The next section introduces the deterministic variant of this problem, where stochastic demand is replaced by deterministic continuous demand. Subsequently two heuristics are presented that are based on the deterministic model, which both are asymptotically optimal.

2.2 Deterministic model

Consider the stochastic model, but now assume that demand in time period t is deterministic with parameters $\lambda(p,t)$ and $\gamma(p,t)$, and relax the integrality constraints. Since the demand functions are regular, the demand can be used as decision variables instead of price, as will be the case. The objective function equals

$$E\left[\sum_{t=1}^{T} r(t,\lambda,\gamma)\right] = \sum_{t=1}^{T} r(t,\lambda,\gamma) = \sum_{t=1}^{T} \big(\lambda(t),\gamma(t)\big)^{\top} p(t,\lambda,\gamma). \tag{6}$$

The second constraint, regarding the distribution of flexibles over specifics, can be relaxed to an inequality without loss of generality. The optimisation problem is therefore given by

$$\max\left\{ \sum_{t=1}^{T} r(t,\lambda(t)\ \gamma(t)) \;\middle|\; \begin{array}{ll} \sum_{t=1}^{T} A\lambda(t) + Bz & \leq C \\ \sum_{t=1}^{T} \gamma(t) - Uz & \leq 0 \\ \lambda,\gamma,z & \geq 0 \end{array} \right\}. \tag{7}$$

To see that the second constraint can be relaxed to an inequality instead of an equality constraint, consider an optimal solution (λ^*,γ^*,z^*) to (7) and assume that the second inequality constraint is strict, i.e., $\sum_{t=1}^{T}\gamma^*(t) < Uz^*$. Construct $\tilde{z}$ by subtracting a total amount of $(Uz^*)_k - \sum_{t=1}^{T}\gamma_k^*(t)$ from the z_i^*-s that correspond to product $k \in F$ in a distributive way, such that $\tilde{z} \geq 0$ and $\sum_i z_i^* - \tilde{z}_i = \mathbf{1}^{\top}\big(Uz^* - \sum_{t=1}^{T}\gamma^*(t)\big)$. Then $U\tilde{z} = \sum_t \gamma^*(t)$ and $\sum_{t=1}^{T} A\lambda^*(t) + B\tilde{z} \leq \sum_{t=1}^{T} A\lambda^*(t) + Bz^* \leq C$, so $(\lambda^*,\gamma^*,\tilde{z})$ is an optimal solution to (7) where the second constraint is tight.

By the regularity assumptions on λ and γ both functions are concave, and therefore the following Karush-Kuhn-Tucker conditions are necessary and sufficient for optimality:

$$
\begin{aligned}
\nabla_\lambda r_\lambda(\lambda,\gamma,t) &= A^\top \pi^*, \quad \forall t,\\
\nabla_\gamma r_\gamma(\lambda,\gamma,t) &= I\rho^*, \quad \forall t,\\
0 &= B^\top \pi^* - U^\top \rho^*\\
(\pi^*)^\top \big(C - \sum_{t=1}^{T} A\lambda(t) - Bz\big) &= 0,\\
(\rho^*)^\top \big(Uz - \sum_{t=1}^{T} \gamma(t)\big) &= 0,\\
\pi^*, \rho^* &\geq 0.
\end{aligned}
\tag{8}
$$

Let $\{\lambda^d(t)\}_{t=1}^T$ and $\{\gamma^d(t)\}_{t=1}^T$ be an optimal solution to the deterministic problem with $\{p^d(t)\}_{t=1}^T$ the corresponding optimal prices and V^d the objective value. The objective function of (7) can be rewritten in a format that is useful in following chapters. Define

$$
\begin{aligned}
\bar{p}_j &= \frac{\sum_{t=1}^T p_j^d(t)\lambda_j^d(t)}{\sum_{t=1}^T \lambda_j^d(t)}, & \alpha_j &= \sum_{t=1}^T \lambda_j^d(t),\\
\bar{p}_k &= \frac{\sum_{t=1}^T p_k^d(t)\gamma_k^d(t)}{\sum_{t=1}^T \gamma_k^d(t)}, & \alpha_k &= \sum_{t=1}^T \gamma_k^d(t).
\end{aligned}
$$

Then the following holds:

$$
V^d = \sum_{t=1}^T \big(\lambda^d(t), \gamma(t)\big)^\top p(t) = \sum_{j\in N} \alpha_j \bar{p}_j^d + \sum_{k\in F} \alpha_k \bar{p}_k^d. \tag{9}
$$

Theorem 2.1 shows that V^d of an optimal solution to the deterministic problem (7) is an upper bound on the objective value V^* of an optimal solution to the stochastic problem (4).

Theorem 2.1. Let V^* be the optimal objective value to (4) and let V^d be the optimal objective value to (7). Then $V^* \leq V^d$.

Proof. Consider the Lagrange relaxation of Equation (4):

$$\begin{aligned}\max_{\lambda,\gamma,z} E\Bigg[& \sum_{t=1}^{T} \big(N(t), M(t)\big)^\top p(t, \lambda, \gamma) \\ & + \pi^\top \left(C - \sum_{t=1}^{T} AN(t) - Bz \right) \\ & + \rho^\top \left(Uz - \sum_{t=1}^{T} M(t) \right) \Bigg],\end{aligned} \tag{10}$$

where $\pi \geq 0$. Consider an optimal solution (λ, γ, z) to the stochastic problem (4). Then it holds that

$$\begin{aligned} C &\geq \sum_{t=1}^{T} AN(t) + Bz \text{ (a.s.)}, \\ 0 &\geq \sum_{t=1}^{T} M(t) - Uz \text{ (a.s.)}. \end{aligned}$$

Since $\pi, \rho \geq 0$, the objective value V^L in the Lagrange relaxation (10) is greater than or equal to the objective value V^* of the stochastic problem (4). Therefore, the objective value $(V^L)^*$ of an optimal solution to the Lagrangian relaxation is greater than or equal to V^*:

$$V^* \leq (V^L)^*.$$

Note that Problem (10) is separable in t, and $E[N(t)] = \lambda(t)$ and $E[M(t)] = \gamma(t)$ hold. Therefore Problem (10) is equivalent to maximising

$$\sum_{t=1}^{T} \left[r(t, \lambda, \gamma) - \pi^\top A\lambda(t) - \rho^\top \gamma(t) \right] + \pi^\top (C - Bz) + \rho^\top Uz.$$

The upper bound holds for all $\pi \geq 0$ and ρ, so also for π^* and ρ^*, the optimal dual prices from the deterministic problem, with corresponding deterministic solution $(\lambda^*, \gamma^*, z^*)$. Complementary slackness gives

$$(\pi^*)^\top (C - A \sum_{t=1}^{T} \lambda^*(t) + Bz^*) = 0,$$
$$Uz^* - \sum_{t=1}^{T} \gamma(t) = 0.$$

Therefore the objective value of the Lagrangian relaxation equals the deterministic objective value. Hence the deterministic solution V^d is an upper bound on the stochastic solution:

$$V^* \leq V^d.$$

□

2.3 Group bookings

Many practical applications allow for group bookings, where customers purchase more than one item of the same product. In the case of specific products, this can be modelled by introducing products with resource consumption lA_j, where $l \in \mathbb{N}$ is the size of the group booking. A flexible product on group products can then easily be defined by defining an appropriate set F_k. The allocation of the flexible product to specific products then proceeds by bulk: all l reservations will be allocated to one specific product j. However, it might be desirable to be able to allocate the l reservations to several *different* specific products. In order to do so, define $Q \in \mathbb{R}^{f \times f}$ such that Q_{ii} is the group-size of product $i \in F$, and $Q_{ij} = 0$ if $i \neq j$. In the stochastic problem (4) the second constraint then changes to

$$Q\sum_{t=1}^{T} M(t) - Uz = 0, \tag{11}$$

and in the deterministic problem (7) the second constraint changes to

$$Q\sum_{t=1}^{T} \gamma(t) - Uz \le 0. \tag{12}$$

The generalisation of the results in this paper the group-booking set-up is straightforward.

3 Solution methods

This section proposes two heuristics to approximate the stochastic problem, called *make-to-stock (MTS)* and *make-to-order (MTO)*. The intuition of both heuristics is based on the MTS and MTO heuristics by Gallego and Van Ryzin (1997).

3.1 Heuristic one: make to stock (MTS)

Let $\{p^d(t)\}_{t=1}^T$ be the optimal deterministic prices. Let $\{\lambda^d(p^d(t),t)\}_{t=1}^T$ and $\{\gamma^d(p^d(t),t)\}_{t=1}^T$ be the corresponding intensities. Define the booking limits $b^\lambda \in \mathbb{N}^n$ and $b^\gamma \in \mathbb{N}^f$ by

$$b_j^\lambda := \lfloor \alpha_j \rfloor$$

$$b_k^\gamma := \lfloor \alpha_k \rfloor$$

Price products according to $\{p^d(t)\}_{t=1}^T$ and sell them until inventories are exhausted, booking limits are reached, or the deadline T is reached. Note that the booking limits guarantee that the capacity constraints will be met. During the selling horizon it is therefore not necessary to keep track of the feasibility. Theorem 3.1 below gives a bound on the performance of MTS.

Theorem 3.1. Define $u_j = \sup\left\{p_j^d(t)\,\middle|\,\lambda^d(t) > 0, \gamma^d(t) > 0, \forall t\right\}$ for all $j \in N \cup F$. Let $N(j)$ be Poisson distributed with parameter α_j, for all $j \in N \cup F$. Let V^{MTS} be the objective value of the strategy that follows from MTS evaluated in the stochastic problem (4). Then the following bound holds:

$$\frac{V^{MTS}}{V^*} \geq 1 - \frac{\sum_{j \in N \cup F} u_j \left(\frac{\sqrt{\alpha_j}}{2} + 1\right)}{\sum_{j \in N \cup F} \alpha_j \bar{p}_j^d}. \tag{13}$$

Proof. For each $j \in N \cup F$, denote with $\{T_j^k\}_{k=1}^{N(j)}$ the time periods where purchases occurred (one purchase per T_j^k). Let R^j be the revenue for product $j \in N \cup F$ under the deterministic solution evaluated in the stochastic problem. Then it holds that

$$\begin{aligned} E[R^j] &= E\left[\sum_{k=1}^{N(j)} p_j^d(T_j^k) - \sum_{k=b_j}^{N(j)} p_j^d(T_j^k)\right] \\ &\geq E\left[\sum_{k=1}^{N(j)} p_j^d(T_j^k)\right] - u_j E\left[(N(j) - b_j)^+\right]. \end{aligned} \tag{14}$$

From Wald's identity it follows that

$$E\left[\sum_{k=1}^{N(j)} p_j^d(T_j^k)\right] = E[N(j)]E[p_j^d(T_j^k)] = \alpha_j \bar{p}_j^d.$$

Similar to Gallego and Van Ryzin (1997) the following inequality is used: for a random variable D with mean μ and standard deviation σ, and for any $d \in \mathbb{R}$, it holds that:

$$E[(D - d)^+] \leq \frac{\sqrt{\sigma^2 + (d - \mu)^2} - (d - \mu)}{2}. \tag{15}$$

From equation (15) it follows that

$$E\left[(N(j)-b_j)^+\right] \le \frac{\sqrt{\alpha_j+(b_j-\alpha_j)^2}-(b_j-\alpha_j)}{2}$$
$$\le \frac{\sqrt{\alpha_j}}{2}+|b_j-\alpha_j|.$$

The fact that $b_j = \lfloor \alpha_j \rfloor$ leads to

$$E\left[(N(j)-b_j)^+\right] \le \frac{\sqrt{\alpha_j}}{2}+1,$$

for the second term of (14). Therefore the expected value of R^j satisfies

$$E[R^j] \ge \alpha_j \bar{p}_j^d - u_j(\sqrt{\alpha_j}/2+1).$$

Using the upper bound $V^* \le V^d = \sum_{j\in N\cup F} \alpha_j p_j^d(t)$ completes the proof:

$$\frac{V^{MTS}}{V^*} \ge \frac{\sum_{j\in N\cup F} \alpha_j \bar{p}_j^d - u_j(\frac{\sqrt{\alpha_j}}{2}+1)}{V^d} = 1 - \frac{\sum_{j\in N\cup F} u_j\left(\frac{\sqrt{\alpha_j}}{2}+1\right)}{\sum_{j\in N\cup F} \alpha_j \bar{p}_j^d}.$$

□

A consequence of Theorem 3.1 is that MTS is asymptotically optimal as demand and capacity go to infinity. To see this, consider the following set-up. For $h \in \mathbb{N}$ define $\lambda^h(p,t) = h\lambda(p,t)$, $\gamma^h(p,t) = h\gamma(p,t)$ and $C^h = hC$ for some fixed $\lambda(p,t)$, $\gamma(p,t)$, C, and T.

Corollary 3.1. Let V_h^{MTS} be the optimal objective value under deterministic model h. Then

$$\lim_{h\to\infty} V_h^{MTS} = V^*.$$

Proof. Let $\pi_h^* = h\pi^*$, $z_h^d = hz^d$, and $\rho_h^* = h\rho^*$ for $z^d(t)$ and shadow prices π^* and ρ^*. The claim is that $(h\lambda^d, h\gamma^d, hz)$ is an optimal solution to (8) for model h under prices p^d. Firstly, the revenue function in model h equals

$$
\begin{aligned}
R^h(\lambda^h, \gamma^h, z^h) &= \sum_{t=1}^{T} \big(\lambda^h(t), \gamma^h(t)\big)^\top p(t, \lambda, \gamma) \\
&= h \sum_{t=1}^{T} \big(\lambda(t), \gamma(t)\big)^\top p(t, \lambda, \gamma) \\
&= hR(\lambda, \gamma, z),
\end{aligned}
$$

hence $\bigtriangledown_{\lambda,\gamma,z} R^h(\lambda^h, \gamma^h, z^h) = h \bigtriangledown_{\lambda,\gamma,z} R(\lambda, \gamma, z)$. On the other hand

$$
\begin{aligned}
A^\top \pi_h^* &= hA^\top \pi^*, \\
I\rho_h^* &= hI\rho^*, \\
B^\top \pi_h^* - U^\top \rho_h^* &= h(B^\top \pi^* - U^\top \rho^*),
\end{aligned}
$$

hence the stationarity KKT conditions are satisfied. Next, observe that

$$
\begin{aligned}
C^h = hC = h(\sum_{t=1}^{T} A\lambda(t) - Bz) &= \sum_{t=1}^{T} A(h\lambda(t)) - B(hz), \\
U(hz) = hUz = \sum_{t=1}^{T} h\gamma(t), &
\end{aligned}
$$

hence the primal feasibility KKT conditions are satisfied. Note that

$$
\alpha_j^h = h \sum_{t=1}^{T} \lambda_j^d(t) = h\alpha_j,
$$

for all $j \in N \cup F$. The bound for model h now equals

$$\begin{aligned}
\frac{V_h^{MTS}}{V^*} &\geq 1 - \frac{\sum_{j \in N \cup F} u_j \left(\frac{\sqrt{h\alpha_j}}{2} + 1 \right)}{\sum_{j \in N \cup F} h\alpha_j \bar{p}_j^d} \\
&= 1 - \frac{\sum_{j \in N \cup F} u_j \left(\frac{\sqrt{\alpha_j}}{2} + 1 \right)}{\sqrt{h} \sum_{j \in N \cup F} \alpha_j \bar{p}_j^d} \\
&= 1 - O(h^{-1/2}),
\end{aligned}$$

which converges to 1 as $h \to \infty$. □

3.2 Heuristic two: make to order (MTO)

The MTO heuristic prices products according to $\{p^d(t)\}_{t=1}^T$. The difference from the MTS heuristic is that no booking limits are applied. Instead, products are offered until selling the product is infeasible. For specific products j this occurs when the state $(s - A_j, y)$ is infeasible, and for flexible products k when $(s, y + e_k)$ is infeasible. Like the MTS heuristic, MTO is asymptotically optimal and a bound for the performance of MTO is given. For convenience, introduce the following notation for resources i and products $j \in N \cup F$:

$$c_{ij} = \begin{cases} a_{ij} & \text{if } j \in N, \\ \max\{a_{ik} \,|k \in F_j\} & \text{if } j \in F. \end{cases}$$

Theorem 3.2. Let $S_i = \{j \in N \cup F \,|c_{ij} > 0\}$ and $\alpha(S_i) = \sum_{j \in S_i} \alpha_j$. Define $v_i = \max\{u_j \,|j \in S_i\}$, $\bar{v}_i = v_i \max\{c_{ij} \,|j \in S_i\}$, and $\hat{p}_i = \sum_{j \in S_i} \bar{p}_j \alpha_j / \alpha(S_i)$. Let V^{MTO} be the objective value of the strategy that follows from MTO evaluated in the stochastic problem (4). Then the following bound holds:

$$\frac{V^{MTO}}{V^*} \geq 1 - \frac{\sum_{i=1}^{m} \bar{v}_i \sqrt{\alpha(S_i)}}{2\sum_{i=1}^{m} \hat{p}_i \alpha(S_i)}. \tag{16}$$

Proof. Consider a modified system where negative inventories are allowed, but a penalty of v_i is charged for every unit of resource i that is backlogged. In this system, if α_{ij} is strictly greater than the capacity left of i, backlogged revenue for product $j \in N \cup F$ is equal to

$$p_j(t) - v_i \leq p_j(t) - u_j \leq 0. \tag{17}$$

Backlogged products also consume resources, which cannot be used by non-backlogged products. Therefore $V^{mod} \leq V^{MTO}$ holds, where V^{mod} is the revenue corresponding to the modified model. In expectation this gives

$$V^{MTO} \geq V^{mod} = \sum_{j \in N \cup F} E[\sum_{k=1}^{N(j)} p_j^d(T_j^k)] - \sum_{i=1}^{m} v^i E[(\sum_{j=1}^{n} c_{ij} N(j) - C_i)^+]. \tag{18}$$

Like in the proof for MTS the first term equals $\sum_{j=1}^{n} \alpha_j \bar{p}_j$. For the second term, observe that if $d \geq \mu$ Equation (15) becomes (triangle inequality)

$$E[(D-d)^+] \leq \frac{\sqrt{\sigma^2 + (d-\mu)^2} - (d-\mu)}{2} \leq \frac{\sigma + (d-\mu) - (d-\mu)}{2} = \frac{\sigma}{2}. \tag{19}$$

Note that

$$E[\sum_{j \in N \cup F} c_{ij} N(j)] = \sum_{j \in N \cup F} c_{ij} \alpha_j \leq C_i.$$

It therefore follows that

$$E[(\sum_{j\in N\cup F} c_{ij}N(j) - C_i)^+] \le \frac{\sigma_i}{2},$$

with

$$\sigma_i^2 = Var[\sum_{j\in N\cup F} c_{ij}N(j)] = \sum_{j\in N\cup F} c_{ij}^2 Var(N(j)) = \sum_{j=1}^{n} c_{ij}^2 \alpha_j$$
$$\le \max\left\{ c_{ij}^2 \middle|\, j \in S_i \right\} \alpha(S_i).$$

Hence

$$\begin{aligned}
\sum_{i=1}^{m} v_i E[(\sum_{j\in N\cup F} c_{ij}N(j) - C_i)^+] &\le \sum_{i=1}^{m} v_i \frac{\sigma_i}{2} \\
&\le \frac{1}{2}\sum_{i=1}^{m} v_i \sqrt{\max\left\{c_{ij}^2 \,|j \in S_i\right\}\alpha(S_i)} \\
&= \frac{1}{2}\sum_{i=1}^{m} v_i \max\left\{c_{ij} \,|j \in S_i\right\}\sqrt{\alpha(S_i)} \\
&= \frac{1}{2}\sum_{i=1}^{m} \bar{v}_i \sqrt{\alpha(S_i)}.
\end{aligned}$$

Hence:

$$\frac{V^{MTO}}{V^*} \ge \frac{V^d - \frac{1}{2}\sum_{i=1}^m \bar{v}_i\sqrt{\alpha(S_i)}}{V^d} = 1 - \frac{\sum_{i=1}^m \bar{v}_i\sqrt{\alpha(S_i)}}{2\sum_{i=1}^m \hat{p}_i\alpha(S_i)}. \qquad (20)$$

□

Corollary 3.2. Let V_h^{MTO} be the optimal objective value under deterministic model h. Then

$$\lim_{h\to\infty} V_h^{MTO} = V^*.$$

Proof. In a similar way as in MTS, for model it holds that $h\ \alpha_h(S_i) = h\alpha(S_i)$ (note that the h-s in $\bar{p}_i$ are cancelled). This gives

$$\frac{V^{MTO}}{V^*} \geq 1 - \frac{\sum_{i=1}^{m} \bar{v}_i \sqrt{h\alpha(S_i)}}{2\sum_{i=1}^{m} \hat{p}_i h\alpha(S_i)} = 1 - O(h^{-1/2}), \tag{21}$$

which goes to 1 as $h \to \infty$. □

4 Numerical Examples

This section provides numerical results to illustrate the model and the performance of the heuristics. The example that is used is based on a problem instance faced by a firm in the online advertisement industry, the problem that motivated the research of this paper. In the online advertisement market publishers sell space on their website to advertisers. Consider a publisher who owns three websites A, B and C. On each website, two advertisement spots are available: a banner (1) on the top of the web page and a box (2) a bit lower. See Figure 1 for a visualisation.

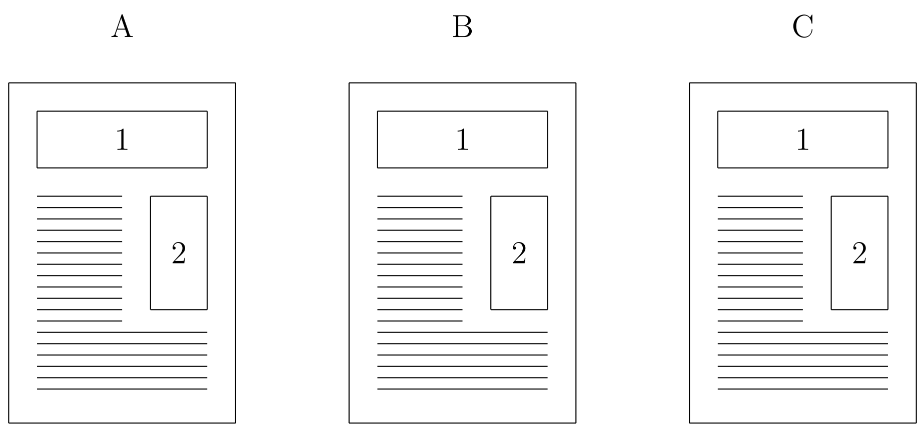

Figure 1: Visualisation of advertisement spots on three websites.

The capacity of the resources are $C = (100, 100, 70, 70, 30, 30)$, which are the views of the websites, respectively. The resources are consumed in $T = 10$ time period (say, weeks, or months), over which the publisher sells them in the form of products. The publisher sells the specific products where only one resource is consumed separately, or both banner and box of one website together. The incidence matrix A is therefore given by

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

The publisher also uses flexible products:

$$F_1 = \{1, 3\}$$
$$F_2 = \{1, 3, 5\}$$
$$F_3 = \{2, 4\}$$
$$F_4 = \{2, 4, 6\}$$
$$F_5 = \{7, 8\}$$
$$F_6 = \{7, 8, 9\}$$

The demand function is linear and independent from other products, i.e., demand for product i is equal to $\lambda_i(t) = a_i(t) + b_i(t)p_i(t)$. The price sensitivity does not change over time, so $b(t) = b$ for all t. However, the intercept $a(t)$ changes over time, and is given by

$$a(t) = \tilde{a}\frac{\sqrt{t}}{\sum_{s=1}^{10}\sqrt{s}}. \tag{23}$$

The parameters $\tilde{a}$ and b are given in Table 1.

		$\tilde{a}_i$	b_i
Specific products	1	259	-0.10
	2	211	-0.09
	3	222	-0.07
	4	114	-0.06
	5	103	-0.05
	6	81	-0.04
	7	168	-0.08
	8	97	-0.07
	9	81	-0.06
Flexible products	1	108.11	-0.04
	2	135	-0.05
	3	108	-0.04
	4	86	-0.06
	5	124	-0.05
	6	92	-0.06

Table 1

4.1 MTS & MTO

First the performance of the MTS and MTO heuristics under the parameters described above is given. To capture the effect of the size of the problem, the problem instances are scaled with a factor $h \in \{0.1, 0.2, 0.5, 1, 2, 5, 10\}$. Furthermore, the heuristics are compared to two additional heuristics: MTS-NF and MTO-NF (No Flexibles). MTS-NF and MTO-NF are exactly the MTS and MTO heuristics, except for the fact that no flexible products are offered. The actual revenue of the strategies is calculated by means of simulation. The errors are within 0.5% of the stated values, with 95% confidence. The results are presented in Table 2.

		MTS		MTO		MTS-NF		MTO-NF	
h	UB	Rev.	%UB	Rev.	%UB	Rev.	%UB	Rev.	%UB
0.1	7869	5372	68.27	6971	88.59	4622	58.74	5352	68.01
0.2	15737	12370	78.60	14548	92.44	10506	66.76	11296	71.78
0.5	39342	34628	88.02	37493	95.30	28471	72.37	29605	75.25
1	78685	72162	91.71	76098	96.71	59133	75.15	60558	76.96
2	157370	148093	94.11	153748	97.70	121330	77.10	123131	78.24
5	393425	379339	96.42	387734	98.55	309386	78.64	312131	79.34
10	786850	767164	97.50	778854	98.98	624898	79.42	628605	79.89

Table 2: Performance of MTS and MTO heuristic, together with MTS-NF and MTO-NF.

The results give two main insights. First, the MTO heuristic performs better than the MTS heuristic in this example. The original problem with $h = 1$ gives an optimality gap of at most 3.29% for MTO and 8.29% for MTS. The gap could be smaller because the reference revenue is only an upper bound on the optimal revenue attainable. For smaller h the optimality gap of MTS increases subtantially to up to 31.73% for $h = 0.1$, while for MTO the gap increases only up to 11.41%. Clearly protecting resources for certain products is not profitable in the case that demand is scarce. As the scale of the problem increases, the optimality gap becomes smaller. The size of the problem compensates the booking limits.
The second insight is that not offering flexible products leads to an enormous revenue loss compared to MTS or MTO: 19.3% for $h = 10$ to up to 23.2% for $h = 0.1$. This is in line with the findings of Gallego et al. (2004), who argue that offering flexible products alongside specific products leads to higher capacity utilization and could attract additional customers without complete cannibalization.

4.2 Estimation error

Uncertainty in demand due to forecasting or estimation errors is an important issue and may lead to suboptimal policies. As an illustration, this example shows the impact of forecast/estimation errors on the performance of the different heuristics. In each simulation the parameters of the assumed demand are randomly drawn according to a normal distribution with mean μ, the parameter value of the true demand ($a(t)$ and $b(t)$), and standard deviation $1.96\sqrt{|\mu|}/l$. Here, l can be interpreted as the sample size, and the standard deviation follows from the confidence interval of the 'estimated' parameter μ. The results are shown in Table 3.

As can be seen from the results, the performance does not suffer that much from forecasting errors. The good performance under demand uncertainty is in line with the results of Petrick et al. (2012). Only for small datasets the revenue loss is 0.9-3.1% for the

		MTS		MTO	
l	UB	Rev.	%UB	Rev.	%UB
∞	78685	72162	91.71	76098	96.71
10	78685	69936	88.88	72479	92.11
20	78685	70918	90.13	73978	94.02
50	78685	71537	90.92	75039	95.37
100	78685	71755	91.19	75469	95.91
200	78685	71875	91.34	75713	96.22
500	78685	72012	91.52	75923	96.49
1000	78685	72065	91.59	76000	96.59

Table 3: Results under estimation errors. The first row, where $l = \infty$, represents the case where demand is known and is the same as in Table 2.

MTS heuristic and 1.4-4.8% for the MTO heuristic (compared to the MTS and MTO heuristics under true demand, respectively). Note that the revenue loss is higher for the MTO heuristic than for the MTS heuristic. This can be explained by the fact that MTS reserves resources for allocation of products. When products are priced too cheap due to forecasting errors, MTO will sell them until the resources are exhausted, such that the more expensive ones can not be sold any more. On the other hand, MTS will protect the resources for the more expensive products.

4.3 Group bookings

In this final example the effect of group bookings is measured. Group bookings are common in many industry applications like hotels, airlines, and online advertisements. In this section the model from Section 2.3 that incorporates group bookings is used. Flexible products are allowed to be sold in groups of size 5 and 10. Demand is based on the parameters of Table 1, except for the fact that the demand parameter $a(t)$ is multiplied by 0.7, for groups of size 5 with 0.2, and groups of size 10 with 0.1. See Table 3 for the results. The MTS and MTO strategies take group bookings into account; MTS-NG and MTO-NG are the MTS and MTO strategies that do not take group bookings into account, but instead price group reservations the same as they do single-product reservations.

		MTS		MTO		MTS-NG		MTO-NG	
h	UB	Rev.	%UB	Rev.	%UB	Rev.	%UB	Rev.	%UB
0.1	4687	2981	63.59	4155	88.64	2934	62.59	4035	86.09
0.2	9374	7177	76.56	8662	92.40	6932	73.94	8403	89.64
0.5	23436	20002	85.35	22323	95.25	21083	89.96	21394	91.29
1	46872	42243	90.12	45328	96.71	42994	91.73	42832	91.38
2	93745	87444	93.28	91567	97.68	85729	91.45	85536	91.24
5	234362	224873	95.95	230919	98.53	213386	91.05	213326	91.02
10	468724	455440	97.17	463773	98.94	426261	90.94	426203	90.93

Three observations can be made from these results. First, there is a clear difference in performance between the MTS and MTS-NG heuristics on one side and the MTO and MTO-NG heuristics on the other side. Similar to previous examples, the MTO heuristic performs better than the MTS heuristic: by 7.3% for $h = 1$ and up to 39.5% for smaller problem sizes. Hence selecting the wrong heuristic can have a dramatic impact on revenue. The fact that MTS reserves resources for products by rounding the deterministic α_j's may cause this significant loss in revenue.

The second observation is that not taking into account group bookings results in a substantial revenue loss of up to 8.1%, for the MTO heuristic. The gap is larger when h increases. MTS, however, does not always outperform MTS-NG: for $h = 0.5$ and $h = 1$ MTS-NG performs 5.4% and 1.8% better than MTS, respectively. According to these examples, MTS does not only perform worse, but also shows no monotonicity in performance, which makes unreliable.

In conclusion, group reservations have a big impact on revenue, but forecasting and implementing a method that can effectively take group reservations into account is challenging.

5 Concluding Remarks

Selling inventory as flexible products in bulk is common practice in the online advertisement industry, which was a motivation for the research topics in this paper. This paper introduces and analyses a pricing-based network RM model with flexible products, and a non-trivial extension to group bookings. Two practical solution methods are described that are efficient to solve and implement: the make-to-order and make-to-stock heuristics, which are based on the deterministic variant of the problem. The analytical results highlight the performance of the heuristics by means of optimality bounds, and it is shown that they are asymptotically optimal as capacity and demand increase. The numerical examples, based on a problem instance from industry, support the analytical findings under different problem sizes.

Besides the flexible nature of the products, the selling in bulk – or in group reservations – is a particular aspect of the problem that is dominating the industry. Group reservations of flexible products offer more flexibility in the assignment of flexible products to inventory, as opposed to specific group reservations. The simulation studies endorse the effectiveness of flexible group reservations.

Another aspect that is important in practice but often overlooked in academics is the robustness of solutions. In practice the actual parameters of the model are unknown and only forecasting estimates are available. The numerical studies in this paper show that under forecast errors the revenue loss is reasonable, implying that the heuristics are quite robust and reliable for applications.

Selecting the correct model, however, is crucial: using a model without flexible products leads to up to 23.2% revenue loss; and not considering group reservations leads to up to 8.1% revenue loss compared to only using single-product reservations.

The results in this paper have several implications that lead to topics for further research. A popular trend in RM assumes demand is given by a choice-model. Solving the problem under choice-based demand is not straightforward, and might lead to promising insights. Also, an in-depth analysis of more heuristics could lead to improved results. Although the optimality gap is at most 3.29% in the case of $h = 1$ in Section 4.1, the potential increase in revenue can lead to substantial increase in profit. The additional revenue will only account for more profit, since no extra costs are incurred.

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