Pricing Derivatives on Multiple Assets: Recombining Multinomial Trees Based on Pascal's Simplex

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Abstract

In this paper a direct generalisation of the recombining binomial tree model by Cox, Ross, and Rubinstein [11] based on the Pascal's simplex is constructed. This discrete model can be used to approximate the prices of derivatives on multiple assets in a Black-Scholes market environment. The generalisation keeps most aspects of the binomial model intact, of which the following are the most important: The direct link to the Pascal's simplex (which specialises to Pascal's triangle in the binomial case); the matching of moments of the (log-transformed) process; convergence to the correct option prices both for European and American options, when the time step length goes to zero and the completeness of the model, at least for sufficiently small time step. The goal of this paper is to present basic theoretical aspects of this approach. However, we also illustrate the approach by a number of example calculations. Further possible developments of this approach are discussed in a final section.

Keywords: Financial Derivative pricing, Multiple assets, Complete Market, Multinomial Trees

1 Introduction

An important aspect of modern economies is the existence of financial markets. On a financial market various assets are traded, such as stocks, bonds, and other financial contracts. Financial contracts which are based on other assets are called *derivatives*. Important classes of derivatives are *European options* and *American options*. The holder of a European contract has the right, but not the obligation, to exercise the option at some pre-specified moment in the future (the maturity date). If the holder exercises the option at that time, he receives a payoff equal to an amount specified in the contract. The holder of an American option is allowed to exercise the option at any time prior to or on the maturity date. An important question in financial mathematics is what a fair price of such options is today.

Many theories and models have been developed to map out the system and behaviour of financial markets. In 1965 Samuelson [23] proposed a popular model for the behaviour of asset prices. In 1973 Black and Scholes [3] provided an equation called the *Black-Scholes equation* to price derivatives on a single asset in the *Black-Scholes model* which describes a Samuelson market environment. In 1985 Cox, Ingersoll, and Ross [10] extended the Black-Scholes equation to the *generalised Black-Scholes equation* to price derivatives on multiple assets. Analytic solutions to these equations have been established for some derivatives, but solving the equation analytically for an arbitrary derivative is usually too hard. Therefore very often discrete methods are applied to approximate the analytically solution.

In 1979 Cox, Ross, and Rubinstein [11] provided a discrete method involving recombining binomial trees based on Pascal's triangle to approximate the price of derivatives on one asset. This paper provides a direct generalisation of this model to a discrete method involving recombining multinomial trees based on Pascal's simplex to approximate the price of derivatives on multiple assets. The path of generalisation that is followed gives new insight in the binomial model as well as the multinomial model. The recombining multinomial trees have a nice structure, which allows them to be used in an efficient way. Moreover, recombining multinomial trees are a useful technique to approximate the value of both European as well as American options.

A decent amount of research has been published on pricing derivatives via discrete approximations. Some of these approximations are tree-based, while others are based on finite-difference methods or Monte-Carlo simulation. Since the focus of this paper is on trees, here only some works related to tree-based methods will be referred to.

In 1979 Cox, Ross, and Rubinstein presented a method to price derivatives on one asset via a recombining binomial tree [11]. A parallel can be made with Pascal's triangle, as is stated in Section 2. In the recombining binomial tree the first and second moment of the continuous time log-transformed process are matched with the first and second moment in the tree corresponding to the discrete time log-transformed process. A no-arbitrage condition is derived and is assumed to hold. The resulting model is complete and the (unique) risk neutral measure (or 'pricing measure' or 'equivalent martingale measure') can be constructed easily.

There are several ways in which the binomial model can be generalised to models for multiple assets. One can distinguish between generalisations in which the discrete-time financial market described is complete, such as in Cheyette [9], He [17] and Chen, Chung, and Yang [8] and which will also be the case in the present paper and generalisations in which the discrete-time financial market described is not complete such as in Boyle [6]; Boyle, Evnine, and Gibbs [7]; Ho, Stapleton, and Subrahmanyam [19].

In Section 2 Pascal's triangle is used to construct binomial trees to price derivatives on a single asset. Our derivation that explicitly uses Pascal's triangle is not known in literature, to the best knowledge of the authors, but is crucial for the generalisation to the multivariate case. A detailed new geometric insight into the structure of the binomial method and the convergence is given, after appropriate scaling, of the probability distribution on the tree to a continuous distribution when the number of steps in the tree increases. The direct link to Pascal's triangle gives rise to the generalisation provided in Section 3, which is based on Pascal's simplex in the case of several risky assets and using a similar geometric projection technique as for the case of one risky asset. These considerations lead to the generalisation of existing methods and to new approaches to efficient calculation of option prices.

As stated, the three articles most related to the present paper are Cheyette [9], He [17] and Chen, Chung, and Yang [8], in chronological order. In Cheyette [9] it was shown how the multivariate geometric Brownian motion with drift model can be approximated by a complete tree, by choosing appropriate direction vectors. However the moments of the tree itself are not matched directly to those of the continuous time model. He [17] showed for a wider class of continuous time models that they can be approximated by complete tree models. He matched the first and second moment of the log-normal process with the first and second moment of his tree, instead of matching the moments of the log-transformed (Gaussian) process with the moments of the log-transformed tree as was the original approach of CRR. As a result He [17] has to assume that the share prices in his tree are non-negative in his specifications and that his trees are free of arbitrage. Chen, Chung, and Yang [8] matches the moments of the log-transformed processes. They focus on the bivariate case and give suggestions as to how to extend this to valuation of options with more than two underlying shares. Furthermore both [17] and [8] assume that the probabilities in the tree are equal, although in the case of [17] this refers to the physical probabilities (resulting in a more complicated formula for the 'risk neutral' probabilities) while in [8] this refers to the risk-neutral probabilities directly.

Other related literature that focuses on using lattice-based methods to price derivatives on multiple assets include the following. All methods have in common that they are not set up in a complete market environment, in contrast to our model. Boyle [6] introduced a tree-based method on two assets with five branches per jump. Boyle, Evnine, and Gibbs [7] generalised this model to k dimensions. One major drawback of this model is that the transition probabilities may be negative. Kamrad and Ritchken [21] improved the efficiency and stability of this model by including more branches. A downside is the computability of such tree with many branches, especially in higher dimensions. Ho, Stapleton and Subrahmanyam [19] expanded the model from [7] with a time-varying variance-covariance structure. However, the model is limited to Bermudan options since the model only allows exercising options at fixed moments in time. Ekvall [12] and Gamba and Trigeorgis [14] introduce improved versions of [7] that avoids the possibility of negative jump probabilities. Recently, Hilliard [18] introduces a simple multivariate grid method to price options, which gives similar results as [12] and [14].

In the present paper we follow the original approach taken by Cox, Ross and Rubinstein closely and stress the link with the Pascal simplex, which appears to be new in this context. In this way the generalisation to the multinomial case is very natural and makes it very easy to understand why and how this should work. In our 'tuning' of the tree we take the variance in the log-transformed tree process equal to the variance of the log-tranformed (Gaussian) continuous-time process, while taking the expectation of the share price process in the tree such that we get a martingale (up to the discount factor). This implies that the probabilities in the tree are the risk-neutral probabilities, so we are modelling directly under the risk-neutral measure (or pricing measure), which is of course the measure required for option pricing and has the advantage that our tree models are arbitrage free by construction. In our trees all share prices are positive. We show that, at least for sufficiently small time step, the tree models represent a complete financial market. This also allows to compute a hedging strategy for a given financial derivative, that is purely based on the tree. From the results of [1] it follows that, under mild conditions, the value of European as well as American options, evaluated using the trees, converge to the corresponding values evaluated using the continuous-time model.

The remainder of this paper is organised as follows. In Section 2 the recombining binomial tree method is derived in a different way than is found in literature. Along this path the generalisation to recombining multinomial trees is presented in Section 3. Some numerical examples are provided in Section 4. Topics for further research are proposed in Section 5. Finally in Section 6 some concluding remarks are made.

2 Recombining Binomial Trees Based on Pascal's Triangle

In 1979 Cox, Ross, and Rubinstein [11] made a major breakthrough by introducing a method to approximate the value of options on a single asset Z via a binomial tree. This section shows that the binomial tree can be constructed using the two-dimensional grid $\mathbb{N}^2$ and Pascal's triangle. The observation that a binomial tree can be interpreted as Pascal's triangle is crucial for the generalisation to the multivariate case in Section 3, where multinomial trees are constructed using Pascal's simplex, the generalisation of Pascal's triangle to higher dimensions.

Consider an asset Z that follows a geometric Brownian motion:

$$dZ = Z\mu dt + Z\sigma dW,$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, and W is a Wiener process.

The goal of this section is to construct a recombining binomial tree with N levels to approximate the price process Z of an asset and the price of a (European) option on this asset. The option is assumed to have a finite expiration date and we consider the asset price and the option price on the finite time interval between the present and the expiration date. The tree can be interpreted as follows. In each time step, the value of the asset can go either up or down (more precisely, the next share price is obtained by multiplication by a factor which can take two values, which we call u and d and all that is required is actually $0 < d < u$), where the asset price at time 0 equals Z_0. Each node in the tree represents a possible outcome of the asset price and has a certain probability to be reached. The tree will be constructed in several steps, starting with the construction of a random walk X on $\mathbb{N}^2$. Using some linear algebra the random walk on $\mathbb{N}^2$ is transformed to a random walk X' on the real line $\mathbb{R}$. Theorem 2.1 below will show that an associated sequence of random variables converges to the asset price process Z at the expiration date. Furthermore, this tree can be used to price the option. The motivation for this construction is linked to Pascal's triangle and the convergence of a sequence of recombining multinomial trees to the asset price process Z.

Two-Dimensional Lattice and Pascal's Triangle

Let $p \in (0,1)$. Consider the two-dimensional Cartesian product $\hat{\mathbb{N}}^2 := \frac{1}{\sqrt{p}}\mathbb{N} \times \frac{1}{\sqrt{1-p}}\mathbb{N}$. Define $\hat{e}_1 := e_1/\sqrt{p} = (1/\sqrt{p}, 0)$ and $\hat{e}_2 := e_2/\sqrt{1-p} = (0, 1/\sqrt{1-p})$. On this lattice, consider a random walk

$$X = \{X_n\}_{n=0}^{\infty} \subset \hat{\mathbb{N}}^2,$$

starting at $X_0 = (0,0) \in \hat{\mathbb{N}}^2$. Each step of the random walk moves 'one unit' in the positive direction of one of the axes. With probability p this is a move in the positive direction of the first axis of $\hat{\mathbb{N}}^2$ ($x \mapsto x + \hat{e}_1$), and with probability $1-p$ it is a move in the positive direction of the second axis of $\hat{\mathbb{N}}^2$ ($x \mapsto x + \hat{e}_2$). The random walk X is thus defined as follows:

$$X_n := \sum_{i=1}^{n} Y_i,$$

where the sequence $Y = \{Y_i\}_{i\in\mathbb{N}}$ is defined by a sequence of i.i.d. random vectors Y_i with

$$P(Y_i = \hat{e}_1) = p,$$
$$P(Y_i = \hat{e}_2) = 1 - p.$$

After N steps there is just a finite number of possible values which X_N can take. Say after N steps there have been i moves in the direction of the first axis and $N - i$ moves in the direction of the second axis, with $0 \leq i \leq N$. Furthermore, there are $\binom{N}{i}$ possible walks which lead to $X_N = i\hat{e}_1 + (N-i)\hat{e}_2$, of which each walk occurs with probability $p^i(1-p)^{N-i}$. Therefore, the probability that $X_N = i\hat{e}_1 + (N-i)\hat{e}_2$ equals

$$P\big(X_N = i\hat{e}_1 + (N-i)\hat{e}_2\big) = \binom{N}{i} p^i (1-p)^{N-i},$$

for all $0 \leq i \leq N$. At level N the value of X_N is equal to $X_N = i\hat{e}_1 + (N-i)\hat{e}_2$ where i has a binomial distribution $B(N,p)$. As a shorthand for this, we will say that X_N has a binomial distribution $B(N,p)$.

The lattice that is constructed in this way is basically Pascal's triangle. After N steps the random walk arrives at the N-th level of Pascal's triangle. See Figure 1 for a visualisation of the construction of this two-dimensional lattice.

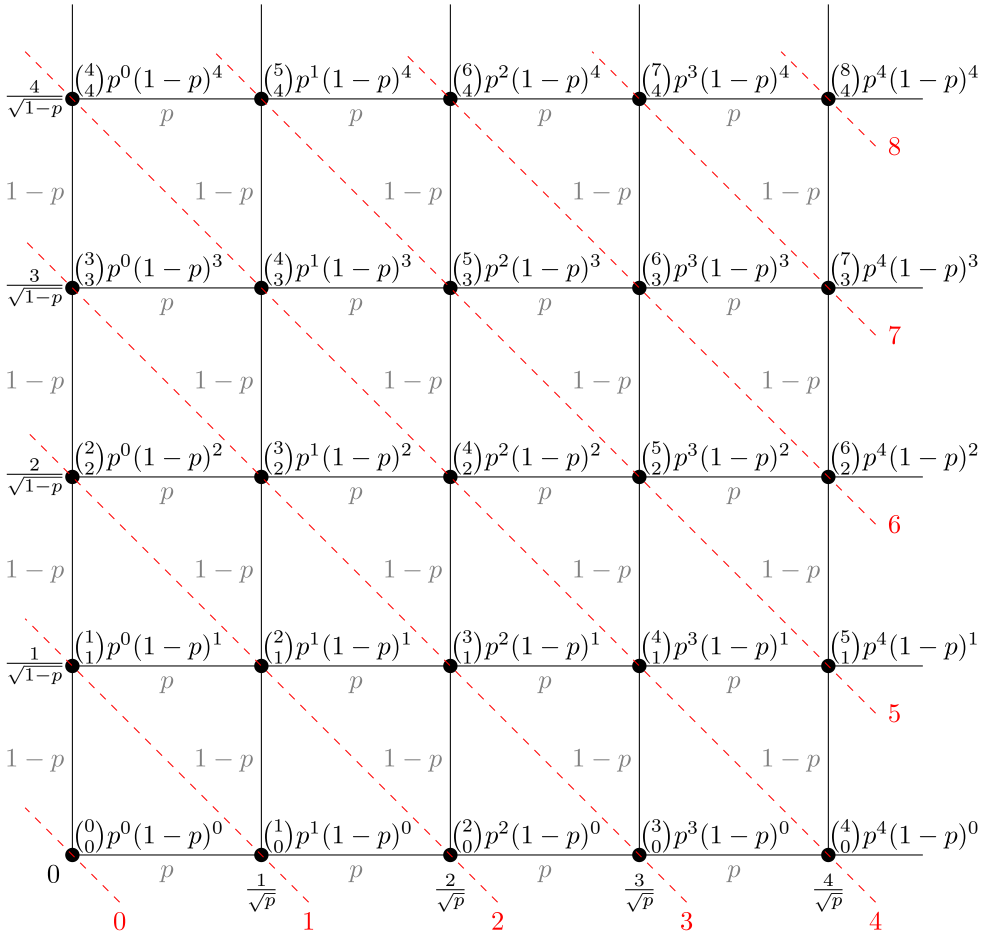

Figure 1: Visualisation of the Pascal-based two-dimensional lattice. Each red dotted line represents the level at which the random walk arrives after n steps, where n is the corresponding number in red.

One-Dimensional Representation of the Two-Dimensional Random Walk

Projecting the random walk X on the hyperplane orthogonal to $\iota = (\sqrt{p}, \sqrt{1-p})$ yields a random walk on the line orthogonal to ι. Now if the whole system is rotated clockwise such that ι coincides with the unit vector $\hat{e}_1$, the projected random walk coincides with a random walk on the second axis of $\mathbb{R}^2$. By eliminating the first axis a random walk X' on $\mathbb{R}$ is constructed.

The random walk X' on $\mathbb{R}$ can also be found in the following way. First the system is rotated such that e_1 coincides with ι. Then the image is reflected on the line through ι. Finally the system is projected on the plane orthogonal to $\hat{e}_1$ through $(0,0)$. This results in a random walk on the hyperplane orthogonal to $\hat{e}_1$ through $(0,0)$. This method is used because it turns out that it works well in the generalisation to multiple dimensions in Section 3. The orthogonal matrix that represents the rotation and reflection is

$$Q = \begin{pmatrix} \sqrt{p} & \sqrt{1-p} \\ \sqrt{1-p} & -\sqrt{p} \end{pmatrix}$$

The matrix that projects the unit vectors on $\mathbb{R}$ is therefore $\tilde{M} = (\sqrt{1-p}, -\sqrt{p})$. By projecting the random walk X on $\mathbb{R}$ via $\tilde{M}$, a new random walk $X' = \{X'_n\}$ on $\mathbb{Z}$ is constructed, defined by

$$X'_n := \tilde{M} X_n,$$

for all $n \geq 0$. Note that X' starts in the origin, i.e., $X'_0 = 0$. In each step, the probability to move up by 'one unit' is equal to p, and the probability to move down by 'one unit' is equal to $1-p$. See Figure 2 for a visualisation of the two-dimensional rotation, and Figure 3 for the projection of the two-dimensional random walk to the one-dimensional walk. In this visualisation a different orientation is used for convenience.

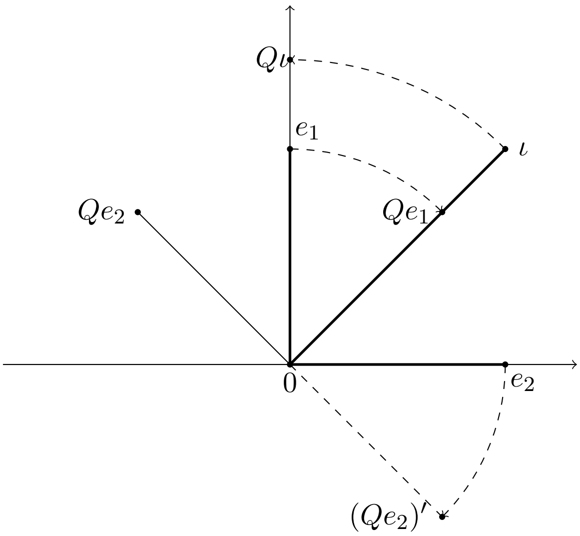

Figure 2: The rotation of the unit vectors and ι by the orthogonal matrix Q. $(Qe_2)'$ is the reflection of Qe_2 in the line through ι.

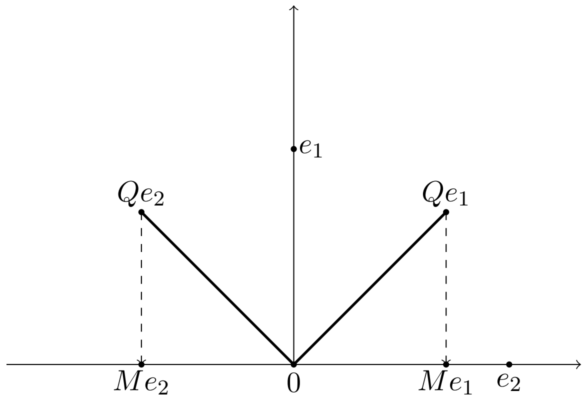

Figure 3: The projection of the rotated unit vectors on $\mathbb{R}$ by M.

Construction of the Recombining Binomial Tree

Let $Y' = \{Y'_n\}_{n\in\mathbb{N}}$ be defined by the projection of Y on $\mathbb{R}$ via $\tilde{M}$, i.e.,

$$Y'_n := \tilde{M} Y_n,$$

for all $n \in \mathbb{N}$. Then Y' consists of a sequence of i.i.d. random variables $\{Y'_n\}_{n\in\mathbb{N}}$ on $\mathbb{R}$, with

$$P\left(Y'_n = \sqrt{\tfrac{1-p}{p}}\right) = p,$$

$$P\left(Y'_n = -\sqrt{\tfrac{p}{1-p}}\right) = 1-p.$$

As a consequence, the expected value of Y'_n is 0, and the variance is equal to 1, for each $n \in \mathbb{N}$. Also for X' the following holds:

$$X'_N = \tilde{M} X_N = \tilde{M} \sum_{n=0}^{N} Y_n = \sum_{n=0}^{N} \tilde{M} Y_n = \sum_{n=0}^{N} Y'_n.$$

Theorem 2.1. Let $\sigma > 0$, $\mu \geq 0$. Then the sequence

$$\{\frac{\sigma}{\sqrt{N}} X'_N + \mu\}_{N \in \mathbb{N}},$$

converges in distribution to the normal distribution $N(\mu, \sigma^2)$.

Proof. Standard, using the Central Limit Theorem. □

The result of Theorem 2.1 is of significant importance for approximating the price process of assets. Consider a price process Z of an asset that follows the geometric Brownian motion given by

$$dZ = Z\mu dt + Z\sigma dW.$$

A classic result of the geometric Brownian motion is that the process $\log Z$ follows the normal distribution $N(\hat{\mu}T, \sigma^2 T)$ on any interval of length T, with $\hat{\mu} = \mu - \frac{1}{2}\sigma^2$ (e.g., see Björk [2], Chapter 5.2, page 65). By applying Theorem 2.1 it can be seen that the price process $\log Z$ on any interval of length T can be approximated by the sequence

$$\{\sigma\sqrt{\delta t} X'_N + \hat{\mu} T\}_{N \in \mathbb{N}},$$

with $\delta t = T/N$. The elements of this sequence can be rewritten such that for each $N \in \mathbb{N}$ it holds that

$$\sigma\sqrt{\delta t} X'_N + \hat{\mu} T = \sum_{n=1}^{N} \left\{ \sigma\sqrt{\delta t} Y'_n + \hat{\mu}\delta t \right\}. \tag{1}$$

Since Y'_n has only two possible outcomes, $\sigma\sqrt{\delta t} Y'_n + \hat{\mu}\delta t$ only has two possible outcomes for each $n \in \mathbb{N}$. This is used to define the *direction vectors* for the log-normal process in Definition 2.1

Definition 2.1. Let the numbers d_1 and d_2 be given by

$$d_1 = \exp\left\{ \sqrt{\tfrac{1-p}{p}}\sigma\sqrt{\delta t} + \hat{\mu}\delta t \right\}, \qquad d_2 = \exp\left\{ -\sqrt{\tfrac{p}{1-p}}\sigma\sqrt{\delta t} + \hat{\mu}\delta t \right\}. \tag{2}$$

Then $d_1 > d_2 > 0$ and we will call d_1 and d_2 the *direction vectors.*

Remark: The reason for this terminology will become clear later on when the multinomial case is treated.

With the direction vectors the log-transformed price process $\hat{Z}$ at time T can thus be approximated by summing the log-transformed asset price at time 0 with $N \in \mathbb{N}$ independent random draws from the distribution χ_N, with χ_N given by

$$P(\chi_N = \log d_1) = p,$$
$$P(\chi_N = \log d_2) = 1 - p.$$

The price process Z at time T on the other hand can be approximated by multiplying the asset price at time 0 with $N \in \mathbb{N}$ independent random draws from the distribution $\chi'_N := \exp(y_N)$, with χ'_N given by

$$P\left(\chi'_N = d_1\right) = p,$$
$$P\left(\chi'_N = d_2\right) = 1 - p.$$

Definition 2.2. Let $N \in \mathbb{N}$. Then the graph containing all possible paths of the random walk with N time steps described above is called a *recombining binomial tree.*

See Figure 4 for a visualisation of a recombining binomial tree with $N = 4$. Note the similarities between the recombining binomial tree in Figure 4 and the two-dimensional lattice in Figure 1. Pascal's triangle can be viewed as the log-transformed tree corresponding to the recombining multinomial tree. Therefore the recombining binomial tree can be viewed as a lattice. This nice structure of the recombining binomial trees implies the potential of the trees to be implemented in an efficient algorithm. Section 5 elaborates on the consequences of this property.

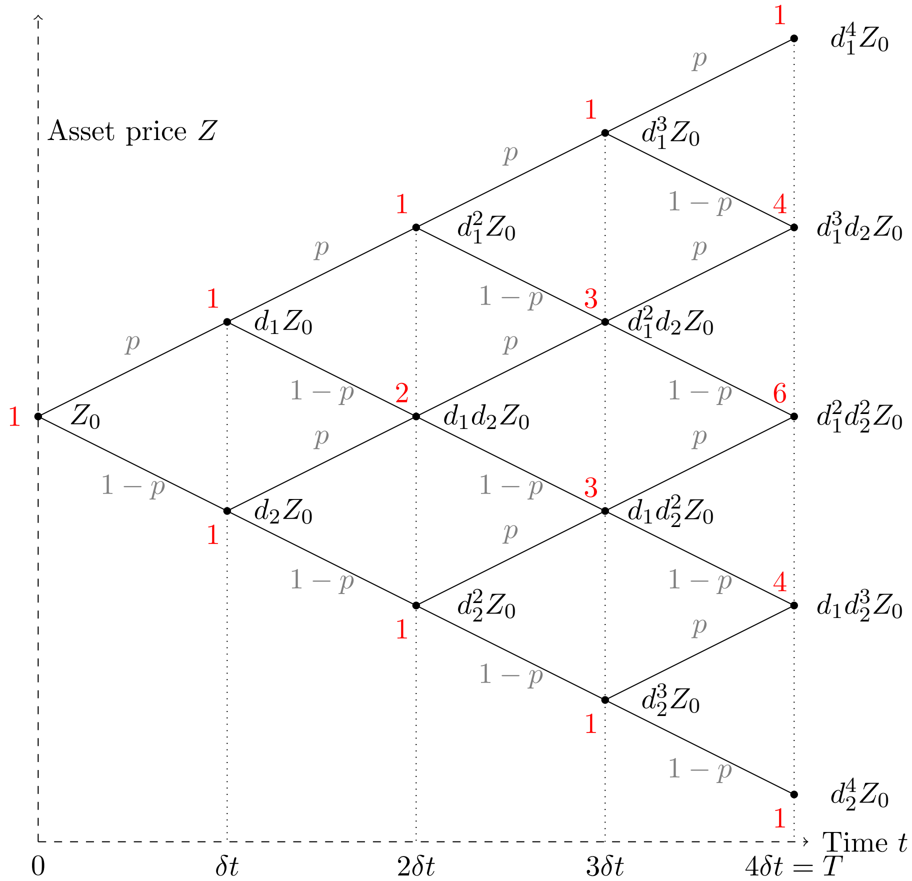

Figure 4: Visualisation of a four-step binomial tree. The red numbers indicate the number of ways to get to the corresponding node.

If the movement is in the first direction d_1 it is called a move *upward*, and if the movement is in the second direction d_2, it is called a move *downward*. In literature the upward move d_1 is often addressed as u, and the downward move d_2 is often addressed as d. The direction vectors

d_1 and d_2 coincide in the binomial case with the direction vectors of the generalisation of the recombining binomial tree to a recombining multinomial tree described in Section 3.

Completeness of the Model and Pricing Derivatives on One Asset

A nice and well known property of the recombining binomial tree is that it is complete under mild conditions, which is stated in Theorem 2.2.

Theorem 2.2. Consider the financial market with one share, with price process described by the binomial tree with direction vectors d_1 and d_2 and associated probabilities $p > 0$ and $1 - p > 0$. Let the interest rate r in this market be constant . If $d_1 > e^{r\delta t} > d_2$ then the market is free of arbitrage and complete.

Proof. Standard; see any textbook on Financial Mathematics treating the binomial tree, such as Björk [2] Proposition 2.26 □

Consider a derivative F with the expiration date at time T, and suppose the value of the derivative $F(Z)$ at time T is known for any Z. Consider the recombining binomial tree with N steps of length $\delta t = T/N$ corresponding to the underlying asset Z of F. Then $F(\nu)$ can be calculated for any node ν in the recombining binomial tree at time T. Suppose that the value $F(\nu')$ is known for all nodes at time $m\delta t$, $1 \leq m \leq N$. Let ν be a node at time $(m-1)\delta t$ and let $\nu_1 = \nu + e_1$ and $\nu_2 = \nu + e_2$. By virtue of Theorem 2.2 a replicating portfolio $\Delta = (\Delta_1, \Delta_2)$ can be found. If F is European, the value of $F(\nu)$ equals

$$F(\nu) = \Delta_1 Z(\nu) + \Delta_2 e^{-r\delta t}.$$

If however F is American, the possibility of an early exercise has to be taken into account. The early exercise leads to a payoff of according to the payoff function F_{payoff}. Therefore the value of the American option $F(\nu)$ equals

$$F(\nu) = \max\left\{F_{\text{payoff}}(\nu), \Delta_1 Z(\nu) + \Delta_2 e^{-r\delta t}\right\}.$$

By working backwards through the recombining binomial tree the value of the option $F(0)$ (American or European) can be found. For the American option the recombining binomial tree has an advantage over other methods: It also shows what the optimal exercise time is. More theory on recombining binomial trees on American options on one asset can be found in Shreve [24] and Shreve [25].

As is well-known the value of the replicating portfolio at node ν can also be computed as the discounted expectation

$$e^{-r\delta t}(qF(\nu + e_1) + (1 - q)F(\nu + e_2)) \tag{3}$$

where $q = \frac{d_1 - e^{r\delta t}}{d_1 - d_2}$ is the 'risk-neutral' (or 'risk-adjusted') probability. For the purpose of option pricing one can consider the continuous-time process under its associated 'risk-neutral' measure $\mathbf{Q}$ which is characterised by the stochastic differential equation

$$dZ = Zrdt + Z\sigma d\bar{W}.$$

where $\bar{W}$ is a standard Brownian motion process under $\mathbf{Q}$. This process can be approximated by a binomial tree with $\mu = r$. The stochastic process described by the tree can be made into a martingale, apart from the usual discounting, by replacing the direction vectors by

$$\tilde{d}_1 = d_1 e^{-\lambda\delta t}, \tilde{d}_2 = d_2 e^{-\lambda\delta t}$$

where λ is chosen such that

$$p\tilde{d}_1 + (1-p)\tilde{d}_2 = e^{r\delta t}$$

hence

$$e^{\lambda\delta t} = (pd_1 + (1-p)d_2)e^{-r\delta t}.$$

Note that λ will converge to zero for $\delta_1 \downarrow 0$, as can be seen by writing

$$d_1 = 1 + \frac{1-p}{p}\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t + \frac{1}{2}\frac{1-p}{p}\sigma^2\delta t + \text{higher order terms}$$

and

$$d_2 = 1 - \frac{p}{1-p}\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t + \frac{1}{2}\frac{p}{1-p}\sigma^2\delta t + \text{higher order terms}$$

which gives

$$pd_1 + (1-p)d_2 - r\delta t = 1 + 0.\sqrt{\delta t} + 0.\delta t + \text{higher order terms}$$

showing that indeed λ will converge to zero if $\delta t \downarrow 0$.

As a consequence using the direction vectors $\tilde{d}_1, \tilde{d}_2$ instead of d_1, d_2 one still gets convergence to the same continuous-time lognormal process for $N \to \infty$. A binomial tree with direction vectors $\tilde{d}_1, \tilde{d}_2$ and probabilities $p > 0$ and $(1-p) > 0$ respectively, is free of arbitrage and complete and the associated risk-neutral probability q is actually equal to p.

3 Recombining Multinomial Trees Based on Pascal's Simplex

Many attempts to generalise the Cox-Ross-Rubinstein model have been made. There are two main ways of generalisations. The first generalises the number of branches, but keeps the number of underlying assets constant, namely equal to one. The second generalises the number of underlying assets from one to many. An example of the first case is the 1986 article by Boyle [5], where a trinomial tree is introduced to price options on one asset. Unfortunately, this model lacked the property of a complete market environment. An example of the second case is the tree-based model to price options depending on two assets Boyle introduced two years later [6]. Again these models are not set up in a complete market environment. In 1990 Cheyette introduced a discrete model that approximates the price of options depending on multiple assets in a complete market environment [9]. In a short paper he showed that the derivative prices derived from a discrete time model converge to the derivative price derived from the generalised Black-Scholes model. A more extensive analysis of a class of models of this type was presented by He [17]. In the present paper it will be shown how the discrete time multivariate generalisation of the recombining binomial tree using Pascal's simplex, the generalisation of Pascal's triangle, can be derived in a very natural fashion. The fact that this model represents a arbitrage-free, complete market under mild conditions is shown. Furthermore, some examples are presented of how to price derivatives depending on multiple assets in Section 4, using the recombining multinomial tree derived in this Section.

$k+1$-Dimensional Lattice and Pascal's Simplex

The recombining binomial tree method in section 2 is based on Pascal's triangle. In the multivariate model the generalisation is based on Pascal's simplex. Pascal's simplex is the generalisation of Pascal's triangle in higher dimensions. The numbers in Pascal's triangle are based on the coefficients of powers of a *binomial*, a polynomial with two terms. The coefficients of the n-th level of Pascal's triangle are the coefficients of the n-th power of the binomial $(x+y)$. The Binomial Theorem states that for all binomials $(x+y)$ and $n \in \mathbb{N}$ the following statement holds:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-1}.$$

The coefficients in the n-th level of Pascal's triangle are therefore given by $\binom{n}{i}$, $0 \leq i \leq n$.

Pascal's simplex is constructed in a similar way in higher dimensions. The elements of Pascals simplex in k dimensions are the coefficients of powers of a *multinomial*, a polynomial with k terms. The Multinomial Theorem states that for any multinomial $(x_1 + \ldots + x_k)$ it holds that

$$(x_1 + \ldots + x_k)^n = \sum_{n_1+\ldots+n_k=n} \binom{n}{n_1, \ldots, n_k} \prod_{i=1}^{k} x_i^{n_i}.$$

The elements in the n-th level of Pascal's simplex therefore consist of the multinomial coefficients

$$\binom{n}{n_1, \ldots, n_k},$$

where $n_1 + \ldots + n_k = n$.

In section 2 Pascal's triangle was constructed on the two-dimensional lattice $\hat{\mathbb{N}}^2$. Let $p \in (0,1)^{k+1}$ with $\sum_{i=1}^{k+1} p_i = 1$. Pascal's simplex can be constructed on a $(k+1)$-dimensional lattice $\hat{\mathbb{N}}^{k+1}$ in a similar way, where $\hat{\mathbb{N}}^{k+1}$ is defined by

$$\hat{\mathbb{N}}^{k+1} := \frac{1}{\sqrt{p_1}}\mathbb{N} \times \ldots \times \frac{1}{\sqrt{p_{k+1}}}\mathbb{N}.$$

Define $\hat{e}_i := e_i/\sqrt{p_i}$. Consider a random walk $X = \{X_n\}_{n\geq 0}$ on $\hat{\mathbb{N}}^{k+1}$, such that in each step a move is made of one step $\hat{e}_i$ in the positive direction of axis i of $\hat{\mathbb{N}}^{k+1}$ with probability p_i. Let $\{Y_n\}_{n\geq 1}$ be a sequence of i.i.d. random vectors in $\hat{\mathbb{N}}^{k+1}$ such that

$$P(Y_n = \hat{e}_i) = p_i,$$

for all $1 \leq i \leq k+1$. Then the random walk X is defined as

$$X_N := \begin{cases} \sum_{n=1}^{N} Y_n & \text{if } N > 0, \\ (0, \ldots, 0) & \text{if } N = 0, \end{cases}$$

for all $N \in \mathbb{N}$. Suppose that after N steps, there have been n_i moves in the direction of the i-th axis, for all $1 \leq i \leq k+1$. There are exactly $\binom{N}{n_1,\ldots,n_{k+1}}$ routes to get to this point, and each route has probability $p_1^{n_1} \cdot \ldots \cdot p_{k+1}^{n_{k+1}}$. The probability to get to $X_N = \sum_{i=1}^{k+1} n_i \hat{e}_i$ therefore equals

$$P\left(X_N = \sum_{i=1}^{k+1} n_i \hat{e}_i\right) = \binom{N}{n_1, \ldots, n_{k+1}} \prod_{i=1}^{k+1} p_i^{n_i}.$$

Derivation of the M-vectors

The movements in $(k+1)$-dimensions need to be translated into a hyperplane of k-dimensions, such that they can be linked to the assets. Therefore the random walk is projected on the hyperplane orthogonal to the vector $\iota = (\sqrt{p_1}, \ldots, \sqrt{p_{k+1}}) = p_1\hat{e}_1 + p_2\hat{e}_2 + \ldots + p_{k+1}\hat{e}_{k+1}$. Since the random walk is defined by the unit vectors $e_i, i = 1, 2, \ldots, k+1$, it is only necessary to consider the projection of these unit vectors. After the projection the $(k+1)$ unit vectors are represented in the k-dimensional hyperplane in $\mathbb{R}^{k+1}$ through the origin orthogonal to ι, but a representation in $\mathbb{R}^k$ is preferred. A rotation can be applied on the whole system such that ι is mapped to the first unit vector $\hat{e}_1$. The $k+1$ rotated projections of the unit vectors $e_1, e_2, \ldots, e_{k+1}$ now lie in a hyperplane orthogonal to $\hat{e}_1$. Therefore these vectors can be represented in $\mathbb{R}^k$ by the projection on the plane orthogonal to $\hat{e}_1$ (the first entries of all $(k+1)$ projected vectors equal 0 and this entry can just be discarded to get the representation in $\mathbb{R}^k$). Define the set of $(k+1)$ vectors thus constructed as the *$\tilde{M}$-vectors.* (see also Definition 3.1 below).

To find the $\tilde{M}$-vectors it is also possible to first apply the rotation, and then project the $(k+1)$ rotated unit vectors on the hyperplane orthogonal to the first unit vector. The difference is subtle, but turns out to be a more efficient calculation. A visualisation of this process for $k = 2$ is given in Figure 5 and Figure 6. Again consider the $(k+1)$ unit vectors $e_1, e_2, \ldots, e_{k+1}$ and ι in $\mathbb{R}^{k+1}$. By applying the correct orthogonal matrix $Q^\top$ to these vectors the system is rotated properly. After the rotation the unit vectors form an orthogonal set of $k+1$ vectors in $\mathbb{R}^{k+1}$. These are the columns of an orthogonal matrix $Q^\top$. By construction $Q^\top\iota = e_1$ and therefore $iota = Qe_1$, so ι is the first column of Q. The other column vectors are arbitrary with length 1 such that Q is orthogonal. This is because rotating $\mathbb{R}^{k+1}$ such that ι us mapped to e_1 can be done in many different ways and the difference between any such rotation is a rotation of the image space that leaves e_1 invariant. Such an orthogonal matrix Q can be constructed by first applying the Gram-Schmidt process to ι and the first k of the $k+1$ unit vectors, of which the last already form an orthonormal basis of $\mathbb{R}^{k+1}$. Note that ι and any k of the $k+1$ unit vectors form a basis of $\mathbb{R}^{k+1}$, because $\iota > 0$ element-wise. Then by normalising the orthogonal basis constructed by the Gram-Schmidt process an orthonormal basis Q is constructed. The result is given below:

$$Q_{ij} = \begin{cases} \sqrt{p_i} & \text{if } j = 1, \\ \sqrt{\dfrac{\sum_{m=i+1}^{k+1} p_m}{\sum_{m=i}^{k+1} p_m}} & \text{if } j = i+1, \\ 0 & \text{if } j > i+1, \\ -\sqrt{\dfrac{p_{j-1}p_i}{\left(\sum_{m=j}^{k+1} p_m\right)\left(\sum_{m=j-1}^{k+1} p_m\right)}} & \text{if } j < i+1. \end{cases}$$

for all $1 \leq i, j \leq k+1$.

The matrix Q is an orthogonal positive lower Hessenberg matrix and $Q^\top$ is an orthogonal positive upper Hessenberg matrix. The general formula for an orthogonal positive upper Hessenberg matrix can be found for example in [16], from which the present formula follows in a straightforward manner (by specialising to the case where the first row is strictly positive).

Consider the orthogonal matrix Q and the unit vectors of $\mathbb{R}^{k+1}$. By applying $Q^\top$ to the unit vectors the rotation gives an orthonormal basis which consists of the column vectors of $Q^\top$, which are the row vectors of the orthogonal matrix Q. For $k = 2$ the matrix $Q^\top$ is given by

$$Q^\top = \begin{pmatrix} \sqrt{p_1} & \sqrt{p_2} & \sqrt{p_3} \\ \sqrt{1-p_1} & -\sqrt{\dfrac{p_1p_2}{1-p_1}} & -\sqrt{\dfrac{p_1p_3}{1-p_1}} \\ 0 & \sqrt{\dfrac{p_3}{1-p_1}} & -\sqrt{\dfrac{p_2}{1-p_1}} \end{pmatrix}.$$

See Figure 5 for a visualisation of the rotation for $k = 2$.

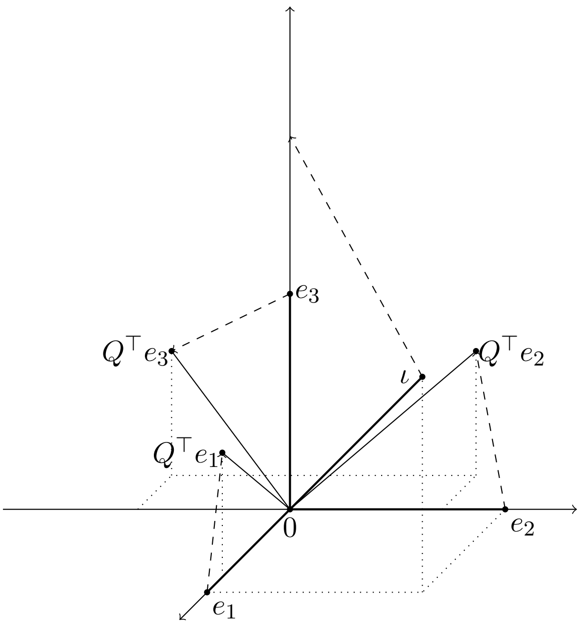

Figure 5: The rotation of the unit vectors and ι by the orthogonal matrix Q.

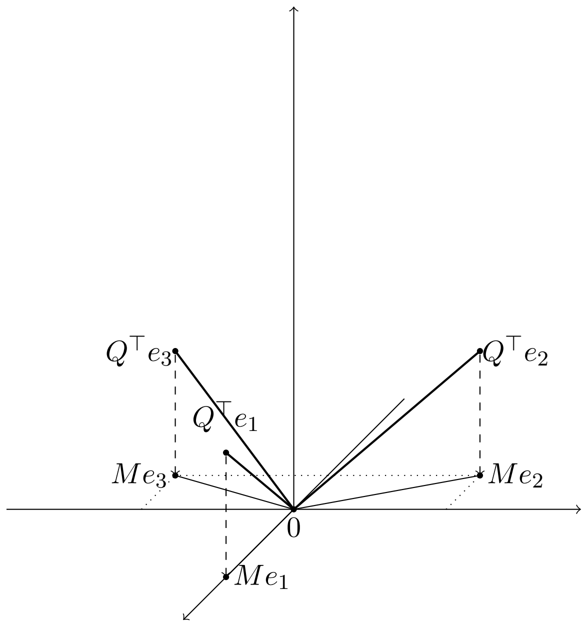

Figure 6: The projection of the rotated unit vectors on $\mathbb{R}$.

The first row of $Q^\top$ is consistent with our requirement that $Q^\top$ is a rotation matrix that rotates ι to the unit vector. The projection of the $(k+1)$ column vectors of $Q^\top$ on the hyperplane orthogonal to e_1 is realised by eliminating the entries in the first row of $Q^\top$. This leads to a $k\times(k+1)$ matrix $\tilde{M}$ with column vectors in $\mathbb{R}^k$. See Figure 6 for a visualisation of the projection of $Q^\top$ on the plane orthogonal to e_1 for $k=2$. From the matrix $\tilde{M}$ we will define the $\tilde{M}$-vectors. Recall that $X_N = \sum_{n=1}^N Y_n$ and that Y_n can take the values $\hat{e}_i = \frac{1}{\sqrt{p_i}} e_i,\ i = 1,2,\ldots,k+1$. So the vectors that $\hat{e}_i,\ i=1,2,\ldots,k+1$ are mapped to are the column vectors of the matrix $\tilde{M}.\mathrm{diag}(\frac{1}{\sqrt{p_1}},\frac{1}{\sqrt{p_2}},\ldots,\frac{1}{\sqrt{p_{k+1}}})$. These columns will be called the M-vectors.

Definition 3.1. The elements of $\tilde{M}$ are given by

$$\tilde{M}_{ij} = \begin{cases} \sqrt{\dfrac{\sum_{m=j+1}^{k+1} p_m}{\sum_{m=j}^{k+1} p_m}} & \text{if } i=j, \\ 0 & \text{if } i>j, \\ -\sqrt{\dfrac{p_i p_j}{\left(\sum_{m=i+1}^{k+1} p_m\right)\left(\sum_{m=i}^{k+1} p_m\right)}} & \text{if } i<j. \end{cases}$$

The column vectors of $\tilde{M}$ are called *$\tilde{M}$-vectors.*. Let $M = \tilde{M}.\mathrm{diag}(\frac{1}{\sqrt{p_1}},\frac{1}{\sqrt{p_2}},\ldots,\frac{1}{\sqrt{p_{k+1}}})$. The columns of this matrix will be called the M-vectors.

Remark: Note that the first row of $Q^\top.\mathrm{diag}(\frac{1}{\sqrt{p_1}},\frac{1}{\sqrt{p_2}},\ldots,\frac{1}{\sqrt{p_{k+1}}})$ has all entries equal to one and that the columns of this matrix are all orthogonal to each other (but do not have length one of course).

Construction of the Recombining Multinomial Tree

The $k \times (k+1)$ matrix that contains both the rotation and projection of the unit vectors in $\mathbb{R}^{k+1}$ to $\mathbb{R}^k$ is equal to $Q^\top$ without it's first row, and therefore is equal to $\tilde{M}$. The random walk X can therefore be projected on $\mathbb{R}^k$ by applying the matrix $\tilde{M}$. This leads to a random walk $X' = \tilde{M}X$ on $\mathbb{R}^k$. As Y_n takes on one of the values $\hat{e}_i = \frac{1}{\sqrt{p_i}}e_i$ in each step a move in the direction of $M_{\bullet i} := Me_i$ occurs with probability p_i, for all $1 \leq i \leq k+1$. Define the sequence $Y' = \{Y_n\}_{n\geq 1}$ by $Y'_n = \tilde{M}Y_n$ for all $n \geq 1$. Then X'_N can be rewritten for all $N \in \mathbb{N}$ as

$$X'_N = \tilde{M}X_N = \tilde{M}\sum_{n=1}^{N} Y_n = \sum_{n=1}^{N} \tilde{M}Y_n = \sum_{n=1}^{N} Y'_n.$$

Theorem 3.1. Let $k \in \mathbb{N}$ and suppose that $\mu \in \mathbb{R}^k_+$ and that Σ is a $k \times k$ symmetric positive definite matrix. Let L be a $k \times k$ matrix such that $\Sigma = LL^\top$. Then the sequence

$$\{\frac{1}{\sqrt{N}}LX'_N + \mu\}_{N\in\mathbb{N}},$$

converges in distribution to the multivariate normal distribution $N_k(\mu, \Sigma)$.

Proof. For $n \in \mathbb{N}$ the expected value of LY'_n is equal to

$$E[LY'_n] = L\tilde{M}E[Y_n] = L\tilde{M}\iota = (0, \dots, 0),$$

where the last equality holds because each row of $\tilde{M}$ is orthogonal to ι. On the other hand, the covariance matrix of LY'_n is equal to

$$\begin{aligned} Cov(LY'_n, LY'_n) &= E[LY'_n(LY'_n)^\top] - E[LY'_n]E[LY'_n]^\top \\ &= L\tilde{M}E[Y_nY_n^\top]\tilde{M}^\top L^\top. \end{aligned}$$

For $E[Y_nY_n^\top]$ it holds that

$$E[Y_nY_n^\top] = \sum_{i=1}^{k+1} p_i\hat{e}_i\hat{e}_i^\top = \sum_{i=1}^{k+1} e_ie_i^\top = I,$$

such that

$$Cov(LY'_n, LY'_n) = L\tilde{M}I\tilde{M}^\top L^\top = \Sigma,$$

for all $n \in \mathbb{N}$. Hence the sequence $\{LY'_n\}_{n\geq 0}$ is a sequence of i.i.d. random vectors with mean 0 and covariance matrix Σ. Also, for each $N \in \mathbb{N}$ it holds that

$$\frac{1}{\sqrt{N}}LX'_N = \frac{1}{\sqrt{N}}L\sum_{n=1}^{N} Y'_n = \sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} LY'_n\right).$$

By using the Multivariate Central Limit Theorem to the sequence $\{LY'_n\}_{n\geq 0}$ of i.i.d. random vectors, it can be seen that the sequence $\{\frac{1}{\sqrt{N}}LX'_N\}_{N\in\mathbb{N}}$ converges in distribution to the multivariate normal distribution $N_k(0, \Sigma)$. By adding the mean μ it follows that the sequence $\{\frac{1}{\sqrt{N}}LX'_N + \mu\}_{N\in\mathbb{N}}$ converges in distribution to the multivariate normal distribution $N_k(\mu, \Sigma)$.

□

For any symmetric positive definite matrix Σ the $k \times k$ matrix L in Theorem 3.1 can be computed by applying *Cholesky decomposition*. In fact, as is well-known, every matrix A which satisfies $AA^\top = \Sigma > 0$ is equal to L post-multiplied by an orthogonal matrix.

Theorem 3.1 can be used to make the connection between random walks on $\mathbb{R}^k$ and a continuous-time price process of k assets. Let Z be the continuous-time price process of k assets that follows a multivariate geometric Brownian motion with drift given by

$$dZ_i = Z_i \mu_i dt + Z_i \sigma_i . dW_i,$$

for all $1 \leq i \leq k$, where $W = (W_1, W_2, \ldots, W_k)^\top$ is a multivariate Brownian motion process with covariance matrix given by $E(W(1)W(1)^\top = \Gamma$; the matrix Γ is chosen such that all its diagonal elements are equal to one (hence Γ is a so-called correlation matrix). As a result each of the processes W_i are standard scalar Brownian motion processes and the vector process $(\sigma_1 W_1, \sigma_2 W_2, \ldots, \sigma_k W_k)^\top$ is a multivariate Brownian motion process with covariance matrix given by $E(W(1)W(1)^\top) =: \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k).\Gamma.\text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k)$.

Let $\hat{Z} = \log Z$ be the element-wise log-transformed price process of Z. Similarly to the one-dimensional case it holds that $\hat{Z}(t)$ follows the k-variate normal distribution $N_k(\tilde{\hat{Z}}(0) + \hat{\mu}t, \Sigma t)$ for any $t > 0$, where the logarithm of a vector denotes the vector of logarithms of the vector entries, and where $\hat{\mu}$ denotes the vector with i–th entry equal to $\mu_i - \frac{1}{2}\sigma_i^2$. Let $\delta t = T/N$. From Theorem 3.1 it follows that the value of $\hat{Z}(T) - \hat{Z}(0)$ can be approximated by the sequence

$$\{\sqrt{\delta t} L X'_N + \hat{\mu} T\}_{N \in \mathbb{N}},$$

where L is a $k \times k$ matrix such that $LL^\top = \Sigma$. The elements of this sequence can be rewritten such that

$$\sqrt{\delta t} L X'_N + \hat{\mu} T = \sum_{n=1}^{N} \left(\sqrt{\delta t} L Y'_N + \hat{\mu} \delta t \right).$$

Definition 3.2. Let $d = \{d_i\}_{i=1}^{k+1} \subset \mathbb{R}^k$ be given by

$$d_i(j) := \exp\left\{ \sqrt{\delta t} (L M_{\bullet i})_j + \hat{\mu}_j \delta t \right\}, \tag{4}$$

for all $1 \leq j \leq k$, for all $1 \leq i \leq k+1$. Then the elements $d(j)$, $1 \leq j \leq k$, of d are called the *direction vectors.*

Using the direction vectors the value of $\hat{Z}(T) - \hat{Z}(0)$ can thus be approximated by summing $N \in \mathbb{N}$ independent random draws from the distribution χ_N, with χ_N given by

$$P(\chi_N = \log d_i) = p_i,$$

for all $1 \leq i \leq k+1$. Therefore the value of $Z(T)/Z(0)$ can be approximated by the product of $N \in \mathbb{N}$ independent random draws from the distribution $\chi'_N := \exp(\chi_N)$, that is,

$$P(\chi'_N = d_i) = p_i,$$

for all $1 \leq i \leq k+1$.

Definition 3.3. Let $N \in \mathbb{N}$. Then the graph containing all possible paths of the random walk with N time steps described above is called a *recombining multinomial tree.*

The visualisation of a recombining trinomial tree is given in Figure 7.

Completeness of the Model and Pricing Derivatives on Multiple Assets

As stated earlier completeness of an arbitrage-free financial market is an important property, which allows one to compute a replicating portfolio for any financial derivative, and hence to compute the value of such a financial derivative. A generalisation of Theorem 2.2 for our multinomial model for sufficiently small time step $\delta t > 0$ is presented below in Theorem 3.2. Consider the direction vectors in the case $\mu_j = r,\ j = 1, 2, \ldots, k$, and corresponding probability vector $p = (p_1, p_2, \ldots, p_{k+1})$. Let $\lambda_j,\ j = 1, 2, \ldots, k$ be the real numbers defined by $p_1 d_1(j) + p_2 d_2(j) + \ldots + p_{k+1} d_{k+1}(j) = e^{-\lambda_j \delta t},\ j = 1, 2, \ldots, k$. Let $\tilde{d}(j) := d(j) e^{(\lambda_j + r)\delta t},\ j = 1, 2, \ldots, k$. We will call the $\tilde{d}_i,\ i = 1, 2, \ldots, k+1$ the *bias-corrected direction vectors*. Note that this correction amounts to replacing $\hat{\mu}_j = r - \frac{1}{2}\sigma_j^2$ by $r - \frac{1}{2}\sigma_j^2 + \lambda_j,\ j = 1, 2. \ldots, k$ in the definition of the direction vectors. Consider the tree with starting value $Z(0)$ and direction vectors $\tilde{d}_i,\ i = 1, 2, \ldots, k+1$ and probabilities $p_1, p_2, \ldots, p_{k+1}$ and time step $\delta t > 0$.

Theorem 3.2. Let $Z(0) > 0, r > 0$ and $\Sigma > 0$ and T be fixed. For each $\delta t > 0$ consider the tree with starting value S_0 and direction vectors $\tilde{d}_i,\ i = 1, 2, \ldots, k+1$ and positive probabilities $p_1, p_2, \ldots, p_{k+1}$. For $\delta t > 0$ sufficiently small the financial market defined by this multinomial tree is complete.

Proof. The market described by the multinomial tree is free of arbitrage, as the probability vector $p = (p_1, p_2, \ldots, p_{k+1})$ is strictly positive and represents the risk-neutral ('martingale') measure. It follows that the market is complete if the matrix of one-step returns of the risk-free bond and the k assets under the $k+1$ possible scenarios is non-singular. As is well-known, for small time step $\delta t > 0$ the returns can be approximated by the logarithmic returns. In this way the question can be reduced to showing that the matrix with first row $(1, 1, \ldots, 1) \in \mathbb{R}^{k+1}$ and the other rows equal to the k rows of LM is non-singular. As L is non-singular, this is equivalent to showing that the matrix with first row $(1, 1, \ldots, 1) \in \mathbb{R}^{k+1}$ and the other rows equal to the k rows of M is non-singular. This matrix can be factored as $Q^\top.\mathrm{diag}(1/\sqrt{p_1}, 1/\sqrt{p_2}, \ldots, 1/\sqrt{p_{k+1}})$, which is obviously non-singular as Q is orthogonal and the probabilities $p_i,\ i = 1, 2, \ldots, k+1$ are positive. $\square$

Consider a derivative F with time to maturity T depending on k assets $Z = (Z_1, \ldots, Z_k)$ and suppose the value of the derivative $F(Z, T)$ is known at time T. Consider the recombining multinomial tree with N steps of length $\delta t = T/N$ corresponding to the underlying assets Z. Then $F(\nu)$ can be calculated for any node ν at time T in the recombining multinomial tree. Suppose that the value of the option is known for all nodes at time $m\delta t$, with $1 \leq m \leq N$. Let ν be a node at time $(m-1)\delta t$ and let $\nu_i = \nu + e_i$ for all $1 \leq i \leq k+1$. By virtue of Theorem 3.2 a replicating portfolio $\Delta = (\Delta_1, \ldots, \Delta_{k+1})$ of F can be found. This replicating portfolio Δ can be found by solving the following system of equations:

$$\begin{pmatrix} Z_1(\nu_1) & Z_2(\nu_1) & \ldots & Z_k(\nu_1) & 1 \\ Z_1(\nu_2) & Z_2(\nu_2) & \ldots & Z_k(\nu_2) & 1 \\ \vdots & \vdots & . & \vdots & \vdots \\ Z_1(\nu_{k+1}) & Z_2(\nu_{k+1}) & \ldots & Z_k(\nu_{k+1}) & 1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{k+1} \end{pmatrix} = \begin{pmatrix} F(\nu_1) \\ F(\nu_2) \\ \vdots \\ F(\nu_{k+1}) \end{pmatrix}. \tag{5}$$

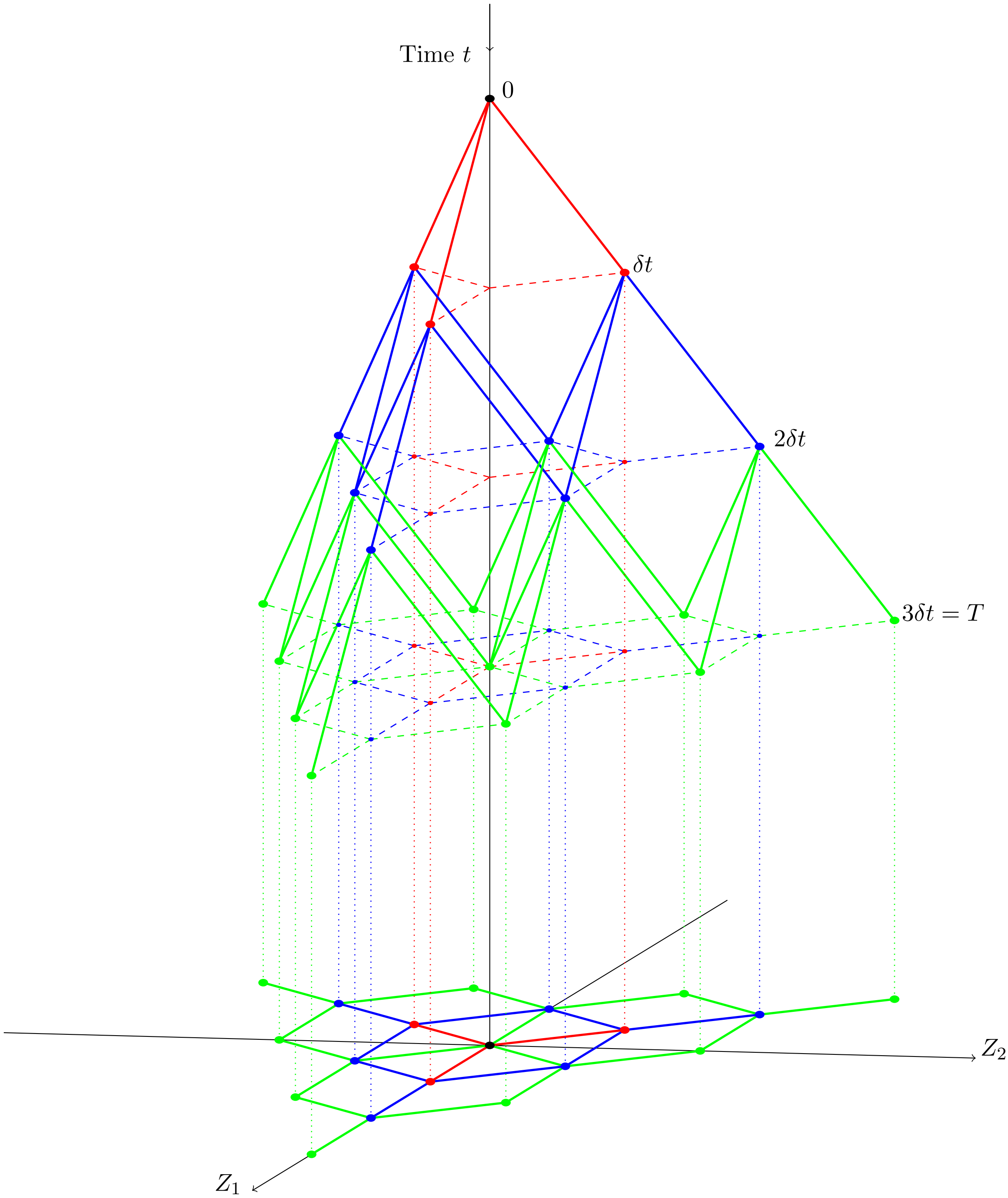

Figure 7: This is a visualisation of a projection of the random walk of the recombining trinomial tree in three dimensions to two dimensions with three time steps of length δt on two assets Z_1 and Z_2 with mean $\mu = (0, 0)$ and $\Sigma = I$. The random walk starts at the black node on top, which is projected on the black node on the two dimensional plane. From there there are three possible moves to time δt, which are represented by the red nodes and lines. The moves from δt to $2\delta t$ are represented by the blue lines, and the possible outcomes are represented by the blue nodes. Finally, the green lines represent the moves from time $2\delta t$ to time $3\delta t$, and the possible outcomes at time $T = 3\delta t$ are represented by the green nodes and the black node.

If F is a European option, the value of $F(\nu)$ equals

$$F(\nu) = \sum_{j=1}^{k} \Delta_j Z_j(\nu) + \Delta_{k+1} e^{-r\delta t}. \tag{6}$$

By working backwards through the recombining multinomial tree the value of $F(0)$ can be found. If however F is an American option, the holder of the option has the choice at node ν to exercise the option and he receives a payoff according to the payoff function $F_{\text{payoff}}(Z(\nu))$. Hence the value of the option at node ν for the American option equals

$$F(\nu) = \max\left\{ F_{\text{payoff}}(Z(\nu)), \sum_{i=1}^{k} \Delta_i Z_i(\nu) + \Delta_{k+1} e^{-r\delta t} \right\}. \tag{7}$$

Just like the two-asset problem discussed in Section 2, the 'risk-neutral' measure can be used to evaluate $F(\nu)$. The direct generalisation of Equation (3) is given by

$$F(\nu) = \sum_{i=1}^{k+1} p_i F(\nu_i) e^{-r\delta t}. \tag{8}$$

To see that this equation is valid, first note that the right hand side of Equation (8) is equal to the right hand side of Equation (5) multiplied by $e^{-r\delta t} p^\top$. This gives

$$e^{-r\delta t} p^\top \begin{pmatrix} F(\nu_1) \\ F(\nu_2) \\ \vdots \\ F(\nu_{k+1}) \end{pmatrix} = p^\top \begin{pmatrix} Z_1(\nu_1) & Z_2(\nu_1) & \dots & Z_k(\nu_1) & 1 \\ Z_1(\nu_2) & Z_2(\nu_2) & \dots & Z_k(\nu_2) & 1 \\ \vdots & \vdots & . & \vdots & \vdots \\ Z_1(\nu_{k+1}) & Z_2(\nu_{k+1}) & \dots & Z_k(\nu_{k+1}) & 1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{k+1} \end{pmatrix} e^{-r\delta t}.$$

Using the definition of the bias-corrected direction vectors, for $j \in \{1, \dots, k\}$ it follows that

$$p^\top \begin{pmatrix} Z_j(\nu_1) \\ \vdots \\ Z_j(\nu_{k+1}) \end{pmatrix} = e^{-r\delta t} Z_j(\nu) \sum_{i=1}^{k+1} p_i \tilde{d}_i(j) = Z_j(\nu). \tag{9}$$

Hence Equation (8) is correct:

$$\sum_{i=1}^{k+1} p_i F(\nu_i) e^{-r\delta t} = \begin{pmatrix} Z_1(\nu) & Z_2(\nu) & \dots & Z_k(\nu) & e^{-r\delta t} \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \vdots \\ \Delta_{k+1} \end{pmatrix} \tag{10}$$

$$= \sum_{j=1}^{k} Z_j(\nu) \Delta_j + \Delta_{k+1} e^{-r\delta t} = F(v). \tag{11}$$

Note that by using the bias-corrected direction vectors, the probability measure given by $p_1, p_2, \dots, p_{k+1}$ on the tree is risk neutral. In the tree without the bias-corrected direction vectors, the first moment and variance of the log share price in the tree model match the corresponding first moment and variance of the log share price in the continuous time model, while in the tree with the bias-corrected direction vectors the first moment of the share price and the variance of the log share price are matching those in the continuous time model.

Convergence of Option Prices

Let a *simple option* be an option that depends on the share prices at the time of exercise but not on the share prices at earlier dates. The recombining multinomial tree method described in this section can be a used to approximate the price of simple European and simple American options. In the case of a simple European option suppose that the pay-off is piecewise continuously differentiable up to the sixth order, with all of these derivatives satisfying a polynomial growth condition (i.e. each one has absolute value that is bounded from above by a polynomial). Then the values of such a European option, as computed using the recombining multinomial trees described in this section converge to the corresponding option value based on the continuous time share-price model when $N \to \infty$. This follows from [17], p.537, condition 1 (the other condition mentioned there is already satisfied by our continuous time model). For the case of simple American options the option values converge to the corresponding option values based on the continuous-time model, if a certain uniform integrability condition is satisfied, namely the condition given on page 301 in formula (5.1) of the paper [1]. In that paper it is argued that for many practical options the uniform integrability condition is indeed satisfied. (Note that the Assumptions 2.1 and 2.2 of that paper are satisfied in our case.) In that same paper also some generalisations are discussed for which the convergence has been proved, for instance for the case that the share pays a continuous dividend yield.

For non-simple options the valuation using the multinomial trees presented here would require generalisations that fall outside the scope of this paper.

4 Numerical Results

This section provides some numerical examples of pricing derivatives on several assets using the theory on recombining multinomial trees discussed in Section 3. The analytical values are also provided, and the results are compared with other models on pricing derivatives on multiple assets. The goal of this paper is to provide a theoretical satisfactory solution rather than a numerical satisfactory solution, and therefore the examples in this section are provided to illustrate the theory. The results show that the estimated option price converges to the analytical value.

In the construction of a tree one needs to decide which probability vector $p \in \mathbb{R}^{k+1}$ and which decomposition matrix L to use, where $\Sigma = LL^\top$. It turns out that different choices lead to different results. Since all trees are valid and the option price derived from the trees converges to the analytical solution, the average value of multiple trees also converges to the analytic solution. In the examples of this section the results of averaging over different trees is considered. Per example there are two parameters that are varied: the depth of the tree N and the number of sample trees $\bar{N}$ that is used for averaging, where $N \in \{1, 2, 5, 10, 20, 50\}$ and $\bar{N} \in \{1, 2, 5, 10, 20, 50, 100, 200, 500, 1000\}$. For each combination of N and $\bar{N}$ the following procedure is repeated 1000 times: construct $\bar{N}$ random trees of depth N, and calculate the corresponding average option price of all $\bar{N}$ trees. A random tree is constructed by drawing an orthogonal matrix Λ uniformly[1] and a random vector p, drawn uniformly with low However, this brings complications to the model. The extent to which this model can be generalised to a market environment with stochastic standard deviations and interest rates has to be investigated. er bound $\underline{p}$[2]. Then $L\Lambda$ is used as the decomposition matrix, since it satisfies $L\Lambda(L\Lambda)^\top = \Sigma$, where L is the Cholesky decomposition of Σ. Over all 1000 repetitions the mean of the option prices is evaluated, as well as the standard deviation.

[1] The Von Misen-Fisher distribution was used to draw orthogonal matrices uniformly

[2] The Matlab function RANDFIXEDSUM.M by Roger Stafford was used to draw the probability vectors.

The example in Section 4.1 discusses an option on two assets to exchange one asset for another. An analytical solution to this problem is provided independently by [13] and [22]. Section 4.2 discusses call and put options on the maximum and minimum of two assets. The analytical solution to this problem is provided by [26]. Finally, Section 4.3 discusses call and put options on the maximum and minimum of three assets, for which the analytical solution is provided by [20].

4.1 Example 1: Exchange Option

Consider a European derivative F with time to maturity T in a Black-Scholes model with two assets. Let the payoff function of F at time T be given by

$$F(Z, T) = \max\{Z_1(T) - Z_2(T), 0\}.$$

The risk-free interest rate equals $r = 0.05$; initial asset prices $Z(0) = (40, 40)$; and covariance matrix equal to

$$\Sigma = \begin{pmatrix} 0.04 & 0.03 \\ 0.03 & 0.09 \end{pmatrix}.$$

The analytical solution to this problem is given by $F^*(Z.0) = 3.219$. The authors of [8] also provided numerical solutions to this problem, and the results are compared to the results stated in their paper. The results are given in Tables 1 and 2.

	$N=5$		$N=10$		$N=20$		$N=50$	
Sample size	mean	sd	mean	sd	mean	sd	mean	sd
1	3.281	0.111	3.249	0.065	3.236	0.045	3.228	0.029
2	3.260	0.079	3.240	0.049	3.229	0.032	3.222	0.019
5	3.273	0.058	3.242	0.033	3.231	0.023	3.223	0.014
10	3.262	0.040	3.234	0.024	3.227	0.015	3.221	0.009
20	3.266	0.031	3.237	0.017	3.229	0.011	3.223	0.007
50	3.266	0.016	3.236	0.010	3.228	0.006	3.222	0.004
100	3.267	0.012	3.237	0.008	3.228	0.005	3.222	0.003
CCY	-		3.264		3.235		3.226	

Table 1: The benchmark results by [8] are denoted by CCY. The analytical solution equals 3.219. Random probability vectors are uniformly drawn with a lower bound of 0.1.

	$N=5$		$N=10$		$N=20$		$N=50$	
Lower bound p	mean	sd	mean	sd	mean	sd	mean	sd
0.001	3.177	0.040	3.197	0.028	3.209	0.019	3.218	0.012
0.01	3.210	0.031	3.224	0.018	3.228	0.011	3.223	0.005
0.1	3.267	0.012	3.237	0.008	3.228	0.005	3.222	0.003
0.2	3.257	0.010	3.239	0.006	3.228	0.004	3.223	0.002
0.3	3.278	0.007	3.239	0.006	3.228	0.003	3.223	0.002
CCY	-		3.264		3.235		3.226	

Table 2: The benchmark results by [8] are denoted by CCY. The analytical solution is 3.219. The results are based on a sample size of 100, for which 1000 simulations ran.

First observe that the results of both Tables 1 and 2 support the theory that the option prices converge as N increases. The analysis of Section 3 shows asymptotic behaviour, but these numerical results show that even for quite small trees the option price can be well approximated. The lower bound on p also influences the mean: the results from Table 2 show that small lower bounds underestimate the analytical solution, while larger upper bounds overestimate the analytical solution. This indicates that the selection of p impacts the outcome. Studies on distributions on p could improve the performance, as will be discussed in Section 5.

A second observation is that the results suggest that the standard deviation heavily depends on the sample size and the lower bound on p. Table 2 shows a small lower bound increases the standard deviation, which can be explained by the increasing size of the set of probability vectors. Table 1 shows that the standard deviation decreases as the sample size increases, as is expected. This implies that using a larger sample size when calculating an option price is desirable to get more stable solutions. Also, as the sample size increases but the depth of the tree remains fixed, the solution seems to converge, but not to the analytical solution. On the other hand, if the sample size remains fixed and the depth of the tree increases, the standard deviation decreases.

4.2 Options on the Maximum and Mimimum of Two assets

This example considers options on the Mimimum and maximum of two assets $Z = (Z_1, Z_2)$. The market environment is the same as in Example 4.1, but now options on the maximum and mimimum of the assets are considered:

1. Call options on the maximum of the assets. Denote this option by $c(K)$, where K is the strike price of the call option. The payoff function at time T equals

$$\max\{0, \max\{Z_1(T), Z_2(T)\} - K\}.$$

2. Put options on the mimimum of the assets. Denote this option by $p(K)$, where K is the strike price of the put option. The payoff function at time T equals

$$\max\{0, K - \min\{Z_1(T), Z_2(T)\}\}.$$

The analytical solution to this problem is presented by [26]. A generalisation of the solution to this problem is presented by [20]. This example has also been used by [6] and [8], to illustrate their tree-based discrete approximations of European options. Their results serve as a benchmark for this example. For both the call and put options we consider all strike prices K with $K \in \{35, 40, 45\}$. In this example, three methods of calculating trees are compared: 1) probability vectors p are fixed at $p = (1/3, 1/3, 1/3)$, but orthogonal matrices are selected randomly (method Q); 2) p is selected randomly, but the decomposition matrix is fixed at L, the Cholesky decomposition (method P); and 3) both p and the decomposition matrices are selected randomly (method P&Q). Based on the results from Example 4.1 a sample size of $\bar{N} = 100$ and a lower bound of $\underline{p} = 0.1$ is used. The results are given in Tables 3 and 4.

The performance of the methods varies: for call options, P seems to underestimate the option value, while Q&P and Q in most cases overestimate the option value. For put options the values for all methods seem closer to each other. However, the standard deviations differ: Q&P has the highest standard deviation in all but one case, and in most cases the standard deviation of P is higher than the standard deviation of Q. From the tree methods, Q and Q&P perform best.

		$N=5$		$N=10$		$N=20$		$N=50$		
K	Met.	mean	sd	mean	sd	mean	sd	mean	sd	AS
	Q	9.482	0.118	9.442	0.096	9.430	0.058	9.422	0.037	
	P	9.354	0.147	9.373	0.078	9.381	0.052	9.394	0.031	
35	Q&P	9.452	0.179	9.431	0.115	9.425	0.076	9.421	0.047	9.420
	B	-		9.404		9.414		9.419		
	CCY	-		9.448		9.220		9.419		
	Q	5.577	0.109	5.521	0.143	5.504	0.072	5.492	0.049	
40	P	5.479	0.167	5.442	0.113	5.451	0.074	5.462	0.045	
	Q&P	5.557	0.206	5.510	0.152	5.501	0.101	5.492	0.063	5.488
	B	-		5.466		5.477		5.483		
	Q	2.784	0.158	2.816	0.095	2.804	0.070	2.795	0.045	
45	P	2.726	0.263	2.741	0.168	2.756	0.116	2.768	0.073	
	Q&P	2.824	0.255	2.811	0.178	2.801	0.122	2.797	0.077	2.795
	B		-	2.817		2.79		2.792		

Table 3: In this table the discrete approximations to price call options $c(K)$ via recombining trinomial trees are stated. $c(K)$ denotes a European call option with strike price K and $p(K)$ denotes a European put option with strike price K. The benchmarks are the results stated by [6], which are denoted by B; and the results stated by [8], which are denoted by CCY. The results are based on a sample size of 100, for which 1000 simulations ran.

		$N=5$		$N=10$		$N=20$		$N=50$		
K	Met.	mean	sd	mean	sd	mean	sd	mean	sd	AS
	Q	1.400	0.078	1.409	0.040	1.395	0.028	1.392	0.019	
35	P	1.404	0.129	1.418	0.070	1.404	0.048	1.398	0.029	1.387
	Q&P	1.401	0.128	1.398	0.078	1.392	0.052	1.389	0.032	
	B	-	-	1.425	-	1.394	-	1.392	-	
	Q	3.891	0.054	3.843	0.083	3.818	0.044	3.806	0.030	
40	P	3.900	0.106	3.848	0.048	3.826	0.025	3.812	0.012	3.798
	Q&P	3.874	0.141	3.830	0.096	3.814	0.063	3.806	0.038	
	B	-	-	3.778	-	3.790	-	3.795		
	Q	7.571	0.119	7.533	0.045	7.513	0.039	7.505	0.022	
45	P	7.584	0.131	7.539	0.087	7.524	0.056	7.513	0.034	7.500
	Q&P	7.552	0.182	7.525	0.108	7.512	0.074	7.506	0.046	
	B	-	-	7.475	-	7.493	-	7.499	-	

Table 4: In this table the discrete approximations to price $p(K)$ via recombining trinomial trees are stated. $p(K)$ denotes a European put option with strike price K. The benchmarks are the results stated by [6], which are denoted by B. The results are based on a sample size of 100, for which 1000 simulations ran.

4.3 Options on the Maximum and Minimum of Three assets

In this example we consider options on the maximum and the mimimum of three assets $Z = (Z_1, Z_2, Z_3)$. The initial asset prices are given by

$$Z(0) = (Z_1(0), Z_2(0), Z_3(0)) = (100, 100, 100).$$

The interest rate is given by $r = 0.1$. The correlation coefficients are given by $\rho_{12} = \rho_{13} = \rho_{23} = 0.5$ and the standard deviations $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are given by $\sigma = (0.2, 0.2, 0.2)$. Hence the covariance matrix is given by

$$\Sigma = \begin{pmatrix} 0.04 & 0.02 & 0.02 \\ 0.02 & 0.04 & 0.02 \\ 0.02 & 0.02 & 0.04 \end{pmatrix}.$$

The considered options are call options and put options with strike price 100 on the maximum or mimimum of the three assets. They have time to maturity equal to $T = 1$ year and an exercise price of $K = 100$. Denote $c(\max)$ as the call option on the maximum of the three assets; $c(\min)$ as the call option on the minimum of the three assets; $p(\max)$ as the put option on the maximum of the three assets; and $p(\min)$ as the put option on the minimum of the three assets.

The option prices are approximated using recombining quadrinomial trees. This example has also been used by [7] and [8]. The results are compared with their results. The analytical solution to the problems can be evaluated using the theory by [20]. Again, a sample size of $\bar{N} = 100$ and a lower bound of $\underline{p} = 0.1$ is used. The results are stated in Table 5.

		Number of time steps (N)				Analytical
	Method	20	40	60	80	solution
	P&Q	22.703	22.689	22.684	22.680	
$c(\max)$	BEG	22.281	22.479	22.544	22.576	22.672
	CCY	22.643	22.660	22.664	22.668	
	P&Q	5.240	5.248	5.253	5.250	
$c(\min)$	BEG	5.226	5.237	5.241	5.243	5.249
	CCY	5.293	5.263	5.258	5.259	
	P&Q	0.937	0.933	0.931	0.932	
$p(\max)$	BEG	0.919	0.925	0.928	0.929	0.936
	CCY	0.945	0.940	0.937	0.937	
	P&Q	7.439	7.422	7.417	7.415	
$c(\min)$	BEG	7.240	7.323	7.350	7.364	7.403
	CCY	7.421	7.407	7.406	7.410	

Table 5: In this table the approximations of options on three assets are presented. $c(\max)$ and $c(\min)$ denote European call options with strike price 100 on the maximum and minimum of the three assets, respectively. $p(\max)$ and $p(\min)$ denote European put options with strike price 100 on the maximum and minimum of the three assets, respectively. The results stated by [7] are denoted by BEG. The results stated by [8] are denoted by CCY. P&Q denote our results.

The results show that our method match the results from [7] and [8]. Even for a small number of time steps our results approximate the analytical solution well.

5 Future Directions

The theory on recombining multinomial trees described in Section 3 can be an efficient tool to price options on multiple assets. There appears to be quite some scope for both theoretical and practical developments.

The results by He [17] and Amin and Khanna [1] for the convergence of option prices does not provide any information about the error of the estimates. However it is nice to know some bounds on how close the results by the recombining multinomial trees are to the actual values. In Section 4 some results on bounds are shown, but these are based on simulation. More knowledge on this subject would clarify the quality of the option values computed with the recombining multinomial tree method.

Another direction to improve the theory on recombining multinomial trees is to generalise the model to a market environment with stochastic standard deviations and interest rate. This market environment is more realistic and therefore leads to more accurate results. In order to model stochastic volatility one finds two approaches in the literature, namely one in which one or more extra stochastic sources are included and one in which this is not the case. It would be interesting to investigate whether the tree methods presented in this paper can be generalised to include stochastic volatility. In the first case one would need a tree of higher dimension, to include the extra stochastic sources. In both cases the quantities modelled by the tree no longer have to represent share prices, but can be interpreted as state vectors, from which the share prices are obtained by applying an appropriate mapping.

We would like to highlight two possible ways in which we think the results based on recombining multinomial trees can be improved. These are treated in the following small subsections.

Efficient Calculations: The Field-Programmable Gate Array (FPGA)

The recombining multinomial tree structure presented here is based on the link with the Pascal simplex and with linear dynamics in the $k-$dimensional lattice $\mathbb{N}^k$. These are well-studied structures in computational algebra and in so-called ND systems theory (see Hanzon and Hazewinkel [15], pages 1-6; and Bleylevens [4]). These links can be exploited in setting up an efficient algorithm for option valuation, using concepts from constructive algebra.These calculations make use of the grid of the M-vectors instead of the usual grids such as $\mathbb{N}^k$. In a recombining multinomial tree for which the moments of the log-normal process are matched the approximation of the option price takes a large number of simple operations. Therefore locating the variables in the physical memory of the computer needed to do the calculation is very time consuming compared to the calculation itself. By allocating the data in the memory in an efficient way the computation time can be reduced.

To exploit even further the fact that the underlying computations are simple and that the main difficulty lies in efficient bookkeeping as well as in the large amounts of data that need to be processed, we are presently investigating usage of FPGAs for speeding up the calculations, especially for deeper trees and for models with more assets. A *field-programmable gate array (FPGA)* is an integrated circuit with programmable logical components. The FPGA allows us to do computations at the deepest hardware level and hence to improve computational speed considerably.

Monte Carlo Simulation

The size of the tree grows exponentially as the depth of the tree increases, so at some point it is physically impossible to evaluate a larger tree within a reasonable amount of time. If more accuracy is needed, other methods need to be used. The numerical results of Section 4 showed that the mean option value derived from random trees provide a better result. The probability vector p and the decomposition L, with $LL^\top = \Sigma$, are randomly chosen in these experiments. This suggests to investigate whether choosing these parameters randomly with well-chosen probability distributions can lead to a Monte Carlo method. Monte Carlo methods have the advantage that, if tuned properly, they converge to the correct value and produce a confidence interval that will contain the correct value with given (high) probability. This is currently under investigation.

6 Concluding Remarks

This paper developed a discrete method to approximate the price of derivatives on multiple assets in a Black-Scholes model. It is not always possible to solve the Black-Scholes equation analytically. For the univariate case the binomial model by Cox, Ross, and Rubinstein is reviewed in Section 2. The binomial model is derived in a different way than is found in literature. This new insight in the derivation of the binomial model served as a foundation to generalise the model to multiple dimensions in Section 3. The goal was to keep most aspects of the binomial model intact, such that the difference between the binomial model and the multinomial model would almost only be the different number of underlying assets. Therefore the properties of the model were kept intact concerning the direct link to Pascal's simplex; the matching of the variance of the log-transformed process (opposed to matching the variance of the log-normal process); and the completeness of the model. The link to Pascal's simplex proved to be very useful. Moreover, on account of this property the recombining multinomial trees have the potential to be used in an efficient algorithm using computer algebra, for example on field-programmable gate arrays. The fact that the recombining multinomial trees are setup in a complete market environment makes the model applicable to all derivatives. Also it can be used to provide hedging strategies and it is setup in an arbitrage free system.

The results of the numerical examples described in Section 4 were compared with other related discrete methods. The examples of the model presented in this paper do not show a preference over other models. The advantages of the model presented in this paper are the relatively low number of nodes in the tree and the simple computations necessary.

Section 5 provided some topics for further research concerning recombining multinomial trees. First, it was argued that recombining multinomial trees have the potential to be used in an efficient algorithm using computer algebra. The fact that the model is based on the grid of Pascal's simplex plays a major role. Applying Monte Carlo theory could be another promising research direction. A further topic for research is extending the model to a market environment with stochastic standard deviations and interest rate. This model is more realistic and an efficient generalisation is therefore desirable.

Altogether the recombining multinomial trees provide a theoretically satisfactory solution. It is a direct generalisation of the recombining binomial trees and provides an alternative to solving the generalised Black-Scholes equation. It is hoped that the insights presented in this paper will lead to new developments that can make the recombining multinomial trees approach into an efficient tool for option valuation in practice.

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