Pricing Derivatives on Multiple Assets

Recombining Multinomial Trees Based on Pascal's Simplex

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Example: Option on Two Assets

Consider two assets $Z = (Z_1, Z_2)$ with $Z(0) = (40, 40)$ and an option on the maximum of the two assets:

$$\max\{\max\{Z_1, Z_2\} - K, 0\}.$$

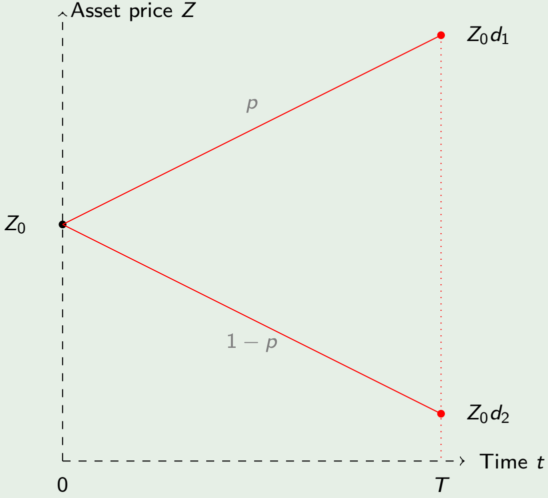
Question

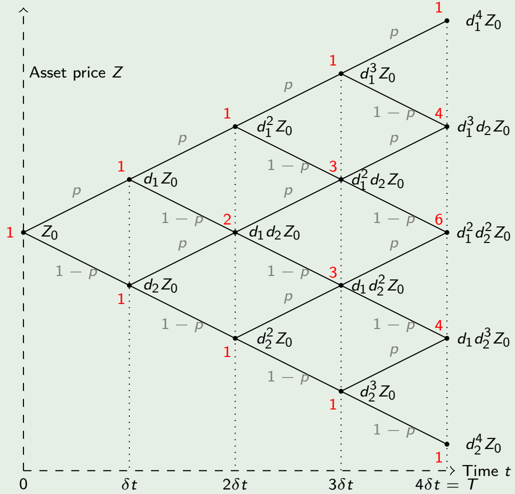
What is a fair price for a derivative?

Solution

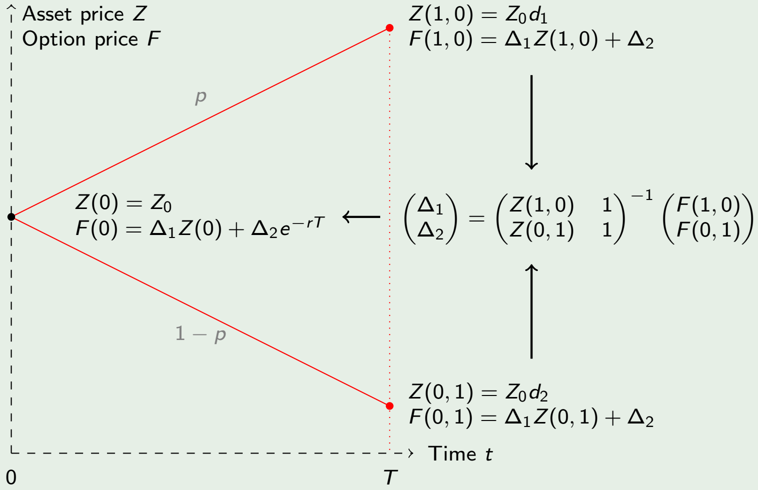
1. Black-Scholes equation (continuous-time)
2. **Recombining Multinomial Trees** (discrete-time)
3. ...

Example: Recombining Binomial Trees




Asset price Z
Time t
0
δt
$2\delta t$
$3\delta t$
$4\delta t = T$
Z_0
$d_1 Z_0$
$d_2 Z_0$
$d_1^2 Z_0$
$d_1 d_2 Z_0$
$d_2^2 Z_0$
$d_1^3 Z_0$
$d_1^2 d_2 Z_0$
$d_1 d_2^2 Z_0$
$d_2^3 Z_0$
$d_1^4 Z_0$
$d_1^3 d_2 Z_0$
$d_1^2 d_2^2 Z_0$
$d_1 d_2^3 Z_0$
$d_2^4 Z_0$
p
$1-p$
1
1
1
1
2
1
1
3
3
1
1
4
6
4
1

Example: Approximate Option Value F



Replicating portfolio: $\Pi(t) = \Delta_1(t)Z(t) + \Delta_2(t)B = F(t)$.

Problem

Find proper values for the direction vectors $d = (d_1, d_2)$.

Solution

Match the first and second moment of the log-transformed process $\log Z$ of the following processes:

- discrete model (recombining binomial tree)
- continuous time model (geometric Brownian motion)

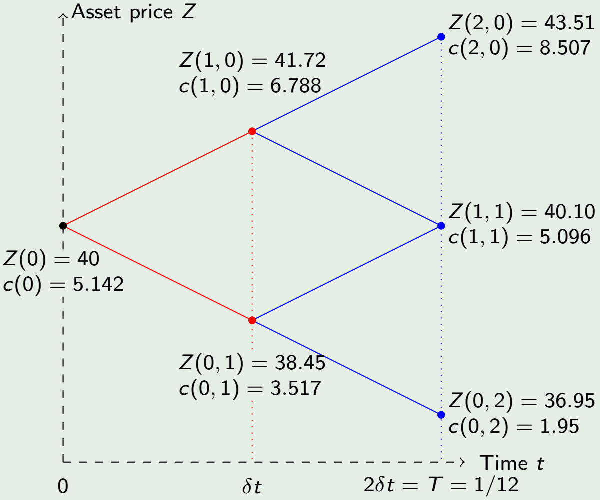
If $\log Z$ follows the normal distribution $N(\mu\delta t, \sigma^2\delta t)$ for any time interval δt, then we choose

$$d_1 = \exp\left\{\sigma\sqrt{\delta t} + \mu\delta t\right\}, \qquad d_2 = \exp\left\{-\sigma\sqrt{\delta t} + \mu\delta t\right\}.$$

Theorem (He 1990, Amin & Khanna 1994)

Let F_n be the option price (European or American) derived from the recombining binomial tree with n timesteps and let F be the analytical value obtained by the Black-Scholes equation. Then

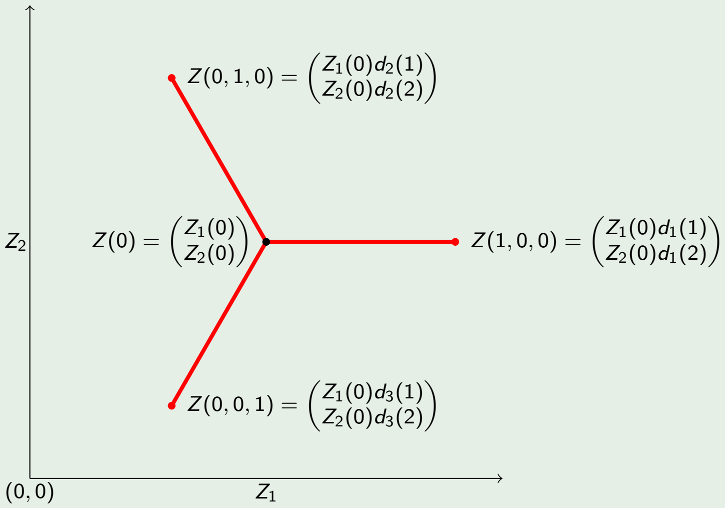
$$\lim_{n \to \infty} F_n = F.$$

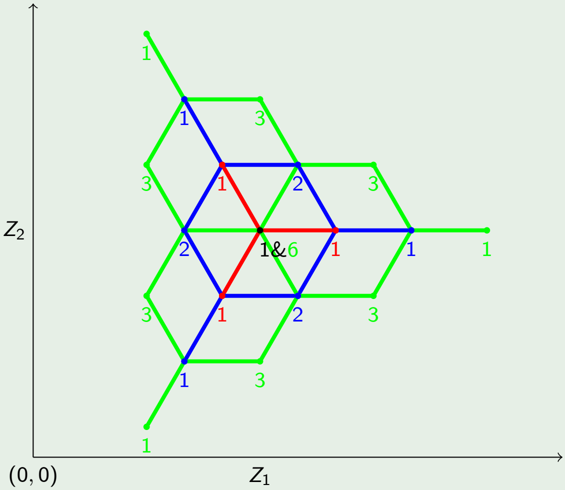

Asset price Z
$Z(2,0) = 43.51$
$c(2,0) = 8.507$
$Z(1,0) = 41.72$
$c(1,0) = 6.788$
$Z(1,1) = 40.10$
$c(1,1) = 5.096$
$Z(0) = 40$
$c(0) = 5.142$
$Z(0,1) = 38.45$
$c(0,1) = 3.517$
$Z(0,2) = 36.95$
$c(0,2) = 1.95$
Time t
0
δt
$2\delta t = T = 1/12$

Example

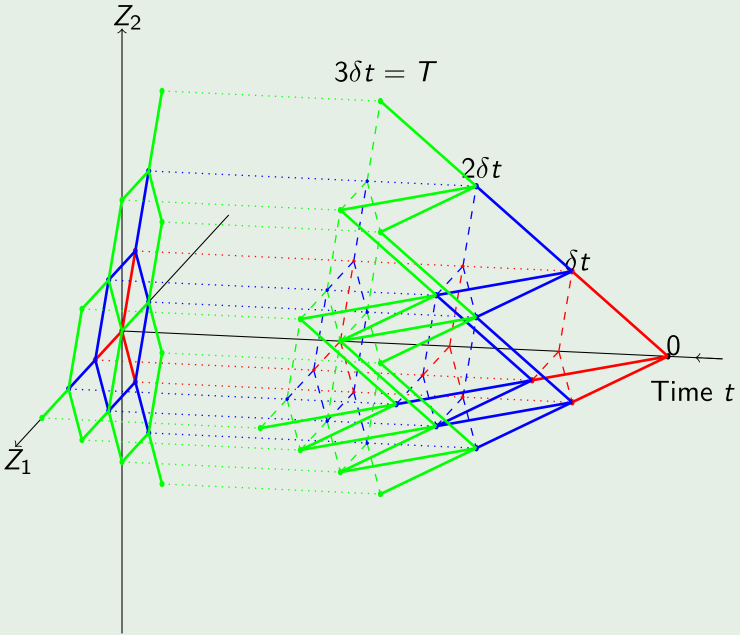
Strike price	Number of timesteps				Analytical
	5	10	20	50	solution
35	5.142	5.147	5.147	5.148	5.148
40	1.049	0.991	0.999	1.003	1.003
45	0.01934	0.01668	0.02168	0.02112	0.02250

Approximation of the value of call options with strike price 35, 40, and 45 and time to maturity $T = 1/12$ on a single asset Z with inital value $Z(0) = 40$, standard deviation $\sigma = 0.2$, and risk-free interest rate $r = 0.05$.


$Z(0,1,0) = \begin{pmatrix} Z_1(0)d_2(1) \\ Z_2(0)d_2(2) \end{pmatrix}$
$Z(0) = \begin{pmatrix} Z_1(0) \\ Z_2(0) \end{pmatrix}$
$Z(1,0,0) = \begin{pmatrix} Z_1(0)d_1(1) \\ Z_2(0)d_1(2) \end{pmatrix}$
$Z(0,0,1) = \begin{pmatrix} Z_1(0)d_3(1) \\ Z_2(0)d_3(2) \end{pmatrix}$
Z_2
$(0,0)$
Z_1



Z_2
Z_1
$(0, 0)$
1
1
3
3
3
2
3
1
1&6
1
1
1
2
3
3
1
2
3
3
1
1
3
1


Z_2
$3\delta t = T$
$2\delta t$
δt
0
Time t
Z_1

Problem

Find proper values for the direction vectors $d = (d_1, d_2, d_3)$.

Solution

If $\log Z$ follows the bivariate normal distribution $N_2(\mu\delta t, \Sigma\delta t)$ for any time time interval δt, then we choose the direction vectors $d = \{d_i\}_{i=1}^3 \subset \mathbb{R}^2$ by (Sierag & Hanzon 2013)

$$d_i(j) := \exp\left\{\sqrt{3\delta t}(LM(\cdot, i))_j + \mu_j\delta t\right\},$$

where L is an 2×2 matrix s.t. $LL^\top = \Sigma$ and M is the 2×3 matrix

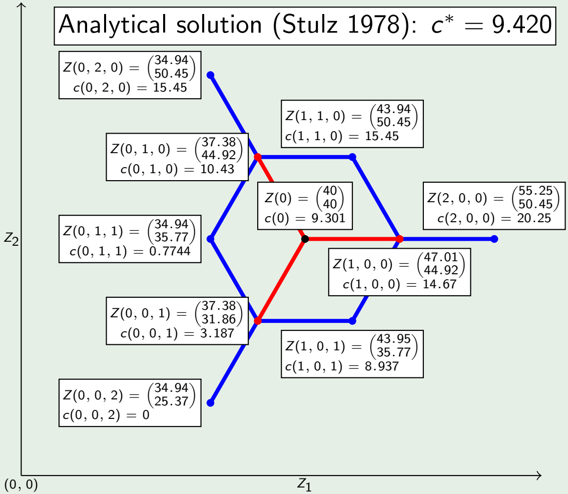
$$M = \begin{pmatrix} \sqrt{2/3} & -1/\sqrt{6} & -1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$

Problem

Approximate the option price via the recombining trinomial tree.

Solution

- Approximate the value of the option at time T for all nodes
- Use argument based on arbitrage-free market to calculate the value of option one timestep prior to T
- Iteratively use this argument to find the value of the option price at time 0


Analytical solution (Stulz 1978): $c^* = 9.420$
$Z(0,2,0) = \binom{34.94}{50.45}$
$c(0,2,0) = 15.45$
$Z(1,1,0) = \binom{43.94}{50.45}$
$c(1,1,0) = 15.45$
$Z(0,1,0) = \binom{37.38}{44.92}$
$c(0,1,0) = 10.43$
$Z(0) = \binom{40}{40}$
$c(0) = 9.301$
$Z(2,0,0) = \binom{55.25}{50.45}$
$c(2,0,0) = 20.25$
$Z(0,1,1) = \binom{34.94}{35.77}$
$c(0,1,1) = 0.7744$
$Z(1,0,0) = \binom{47.01}{44.92}$
$c(1,0,0) = 14.67$
$Z(0,0,1) = \binom{37.38}{31.86}$
$c(0,0,1) = 3.187$
$Z(1,0,1) = \binom{43.95}{35.77}$
$c(1,0,1) = 8.937$
$Z(0,0,2) = \binom{34.94}{25.37}$
$c(0,0,2) = 0$
Z_2
(0, 0)
Z_1

Theorem (He 1990, Amin & Khanna 1994)

Let F_n be the option price (European or American) derived from the recombining trinomial tree with n timesteps and let F be the analytical value obtained by the Black-Scholes equation. Then

$$\lim_{n \to \infty} F_n = F.$$

Generalisation to k Assets

This method can be generalised to derivatives on k assets

- By virtue of the multivariate Central Limit Theorem the multinomial trees approximate the normally distributed log-transformed price process;
- By virtue of He (1990) and Amin & Khanna (1994) the option price derived from the recombining multinomial trees converges to the analytical value.

Example: Numerical Results

Steps (M)	Number of timesteps (N): 1	2	5	10	20	50	100
1	9.599	9.832	9.531	9.561	9.514	9.448	9.425
2	10.085	9.574	9.480	9.333	9.501	9.441	9.421
5	9.908	9.574	9.507	9.465	9.422	9.446	9.424
10	9.641	9.542	9.438	9.493	9.422	9.432	9.414
20	10.050	9.551	9.436	9.464	9.419	9.428	9.426
50	9.689	9.521	9.484	9.435	9.429	9.427	9.419
100	9.686	9.584	9.476	9.431	9.422	9.425	9.423
200	9.715	9.547	9.475	9.446	9.430	9.422	9.419
500	9.732	9.569	9.476	9.450	9.427	9.420	9.419
1000	9.732	9.563	9.475	9.438	9.431	9.421	9.420

Approximation of the price of the option with payout $\max\{\max\{Z_1, Z_2\} - 35, 0\}$. Analytical solution: 9.420.

Further Research: Implementation on FPGA

N	PC	FPGA
250	2.4	0.01317
350	5.7	0.03604
450	14	0.07644
500	43	0.10479

Computation time (seconds) of option on two assets.

Thanks to Christian Spagnol (UCC) for this table.

- FPGA: Field-programmable gate array
- Implementation of the recombining multinomial trees into an efficient algorithm on FPGA
- First results are promising